CHAPTER 5

DIFFERENCES BETWEEN LARGE AND SMALL, GIVEN APPRECIATING AND DEPRECIATING CAPITAL, AND VARIATIONS IN LAND QUALITY

In the real world, production doesn't happen "instantaneously" as assumed in the first four chapters. Rather, it usually happens in cycles, which may be as short as the few minutes to fry a hamburger in a fast food joint, as long as the decades between planting and cutting a tree, or as long as the life of a Roman aquaduct. Over that cycle, assets may appreciate like the tree, or depreciate, like the aquaduct.

How do richer and poorer people, larger and smaller firms-- "Large" and "Small" for short--behave given such possibilities? How does their behavior differ if they all occupy the same quality land, as assumed in Chapters 1, 2 and 4? How does it differ if they can occupy different quality land according to their comparative advantages, as shown in Chapter 3?

Sec. 5.1 describes the models and basic results of Chp. 5. Sec. 5.2 suggests some broader implications.

5.1 Models and Basic Results[A]

Sec. 5.3 presents the basic "point input--point output" model of the appreciating asset, like trees in the forest. Sec. 5.4 presents the "point input--continuous output" model of the depreciating asset, such as buildings in a city.

In both models, landowners determine the optimal life cycle of their trees or buildings by maximizing the present value of their land. This optimal life cycle, (given a wage and discount rate), is an intrinsic property of the production function, just like the intrinsic labor-intensity, as defined in Chp. 3. Sec. 5.5 shows why Large may have a comparative advantage in activities with intrinsically longer cycles.

The models show how Large and Small differ in a number of economic measures (Table 1), notably length of cycle, gross income per acre, profit per acre, average product of labor, labor share of output, and capital turnover. The measures are calculated under two polar assumptions, with two subcases each :

1. Large and Small occupy the same quality land.

   a. The differences of Large and Small are measured by an outside observer who imputes the same wage and discount rate to both.

   b. The differences of Large and Small are measured in terms of their own internal wage and discount rate.

2. Large occupies so much better quality land than Small that quality differences swamp differences due to wage and discount rate.

   a. Better quality land yields the same output with less labor.

   b. Better quality land produces more with the same labor.

There's a good practical reason for comparing effects of the two polar assumptions: In any real world empirical work, it may be very difficult to measure the difference in quality of resources owned by richer or poorer people, larger or smaller firms. So it's important to know which differences, like cycle length, are sensitive to resource quality; and which, like average product of labor, are not sensitive.

Perspective--observer's or owner's--can make a difference too. For example, when an observer measures the value of property, (land plus improvements like trees or buildings), he implicitly or explicitly assumes some average or "market" wage and discount rate. The owner of property measures value by his own internal wage and discount rate--

different for Large and Small. As a result, it appears to an observer that most landowners, Large and Small, do not maximize present value (or profit), --because they do not use the cycle length that is "correct" for the wage and discount rate he imputes.

Empirical studies of differences between Large and Small generally ignore perspective. They may even mix internal and external perspectives, for example measuring property value at "market", and labor costs by the actual wage bill. So it helps the interpretation of data to know which differences hold regardless of perspective, and which do not.

Principal Results:

1. The major results of earlier chapters still hold: As in Chp. 3, Large has a comparative advantage in owning "better quality" land, that is, land where production is less intrinsically labor intensive. Such land is of course more valuable per acre. Regardless of land quality, Large always shows a higher average product of labor. By external measures, and except in one odd case by internal measures too, Large shows lower capital turnover--gross income divided by value of property. It's of course well-documented that average product of labor rises, and capital turnover falls with firm size.

2. By external measure on the same quality land, and in general on better quality land, Large enjoys a higher profit share of income. It's well-documented that profit share of income does in fact rise with firm size. This higher "profitability" of bigger firms is usually attributed either to monopoly profits, or, at the University of Chicago, to greater "efficiency". In fact, higher profitability may merely signal greater capital-intensity.

3. For both trees and buildings, Large uses a longer cycle of

production than Small, on the same quality land. Assuming staggered production, this means that Large's trees or buildings are older on the average. However, for a given wage and discount rate, the better the quality of land, the shorter the cycle. Hence, if Large owns very much better quality land than Small, Large may in fact use a shorter cycle than Small. So it is impossible to predict whether Large uses a longer or shorter cycle than Small, unless they demonstrably occupy the same quality land--such as identical, adjoining property.

4. Large and Small do not differ consistently in other economic measures, unless they occupy the same quality land. Thus, as in earlier models, Large generally gets lower gross income per acre--output per cycle divided by cycle length--on the same quality land, but higher gross income per acre on much better land. Also as in earlier models, Large shows a higher labor share of output on the same quality land, but a lower labor share on much better land.

5. The tree and building models differ in few, but significant, ways. For example, on the same quality land, Large owns a higher ratio of trees (appreciating asset) to land by value, but a lower ratio of buildings (depreciating asset) to land by value, as measured by an outside observer. On much better land, the ratio of improvement to land value is always lower, for trees or buildings. So, to an observer, Large always shows a lower ratio of depreciating assets to land.

There's one curious circumstance where Large may get higher instead of lower gross income per acre on the same quality land. In the tree model for low labor costs and a long enough cycle to make interest costs quite important, gross income may rise a bit before falling as cycle lengthens. That is, there may exist a region of increasing returns to cycle length. For example, imagine that Small cuts a forest for firewood while Large cuts it less often for lumber. The output may be so much more valuable as lumber that gross income increases up to a point as cycle lengthens. (I doubt this ever really happens.)

6. The _way_ in which better land is better makes a difference in some cases. For example, if better land requires less labor for given output, labor per acre and labor cost per acre fall as quality improves. If better land yields more output for given labor, labor per acre and labor cost per acre may rise as quality improves. Thus, although on the same quality land, Large uses less labor, but pays or impute a higher labor _cost_--on better quality land, Large may use less or more labor, and pay less or more for it.

Table 5.1 summarizes results for all measures of differences between Large and Small.

7. For convenience, the tree and building models allow only one "current" input: labor. (The cost of a "current" input, like labor, appears on a firm's income statement; while the cost of an "investment", like a land purchase, appears on a firm's balance sheet). All the results follow from the assumption that, due to transactions costs, Large pays or imputes a higher wage. But the results still hold allowing other current inputs like materials, provided that on the average, Large pays or imputes a higher price for all current inputs including labor. To assume otherwise would violate the basic assumption in Chp. 1, that transactions costs ultimately outweigh any economies of scale (like bulk discounts), creating net diseconomies. So the predictions of the models can be tested on data from real life firms.

8. For very short cycles of production, the tree and building models

become identical to each other and to the "instantaneous" production, profit-maximizing models in previous chapters. If production is not essentially instantaneous, but cycle length times discount rate is very small (much less than one), then profit-maximization still gives the optimum cycle length, but the models differ from each other and from instantaneous production models. In other words, unsurprisingly, profit-maximization closely approximates present value-maximization if very little interest accumulates during a production cycle--true for small cycle length times discount rate. But if cycle length times discount rate is not small, profit-maximization gives too long a cycle for trees, and too short a cycle for buildings--compared to the correct cycle given by present value-maximization.

Table 5.1
(See Tables 5.3 & 5.4 for Derivations)

	Economic Measure	TREE MODEL				BUILDING MODEL			
		Grtr Wealth		Better Land		Grtr Wealth		Better Land	
		Observer	Owner	Lowr Labr	Highr Prod	Observer	Owner	Lowr Labr	Highr Prod
1.	Cycle length: z	+	+	-	-	+	+	-	-
2.	Cycle x discount: rz	+	- mstly*	-	-	+	- mstly	-	-
3.	Output/cycle OP	+	+	-	+	+	+	-	+
4.	Gross income/acre: Y = OP/z	+ ir† - dr	+ ir - dr	- ir + dr	+	-	-	+	+
5.	Labor/acre: L	-	-	- mstly	+	-	-	-	+
6.	Labor cost/acre: wL	-	+ mstly	-	+	-	+	-	+
7.	Rent/acre: R = rV	0	-	+	+	0	-	+	+
8.	Profit/acre: P=Y-wL	+th-**	-	+	+	+ th-	-	+	+
9.	Av.prod.labor:AP=Y/L	+	+	+	+	+	+	+	+
10.	Labor share: LS=wL/Y	-	+	-	-	-	+	-	-
11.	Rent share: RS=R/Y	- ir + dr	-	+	+	+	-	+	+
12.	Profit share:PS= P/Y	+	-	+	+	+	-	+	+
13.	Land value/acre: V	0	0	+	+	0	0	+	+
14.	Total val/acre:W=P/r	+	+ mstly	+	+	-	+ mstly	+	+
15.	Impr.val/acre:IM=W-V	+	"	- or?	+	-	"	-	?
16.	Ratio IM/V	+	"	-	-	-	"	-	-
17.	Capital turnover: TN = Y/W = r/PS	-	- mstly	-	-	-	-	-	-

* "mostly" -- see text and Table 5.3 or 5.4 for explanation.
† Increasing returns and decreasing returns to time. See text & Table 5.3
** "+ then -" as cycle length goes from min to max. See text & Tables 5.3 & 5.4.

5.2 Broader Implications[A]

The tree and building models in fact apply to a wide variety of activities. Hence they predict differences in behavior of richer and poorer people, larger and smaller firms in many different circumstances:

Applications of the Tree Model for Appreciating Assets:

The tree model applies at least roughly to any production process that results in batches of goods which increase in value with time until sold or used at the end of a cycle. The cycle may be intrinsically long, as for trees, or intrinsically short, as for baked goods.

Wine aging in a cellar is another familiar example of goods produced on a long cycle. The cellar owner again maximizes the present value of land: space in his cellar. For cellar space, like forest land, is the limited resource to which the owner imputes rent. New wine can be laid down to age only when the old wine has been sold.

Manufactured goods also fit the model. In most cases, producing goods on a longer cycle increases their quality and value, to a point. (The workmen aren't so rushed; the first coat of paint can dry before the second is applied, and so on). Again, the owner maximizes the present value of scarce factory space.

And inventory held for retail also fits the model. Of course, most inventory does not increase in quality while the retailer holds it. But up to a point, the price the retailer can get increases with the time he holds the inventory. This happens simply because it takes time to make sales. The retailer must wait for customers to come by; the higher his prices, the fewer come, and the less they buy. So the value of a batch of goods can be written as an increasing function of the time they remain in inventory (until they significantly deteriorate). The retailer sets his prices to maximize the present value of limited shelf and storage space, thus choosing the rate his inventory turns over.

So the tree model suggests that, holding constant the quality of the location, richer people and bigger companies age wines longer, produce better quality goods, and sell equivalent goods for higher prices while carrying longer inventories. Not holding constant the quality of location, this contrast may not hold. For in more valuable locations, it pays to speed up the cycle, replacing the wine more often, cranking out goods faster, and turning over inventories faster.

Applications of the Building Model for Depreciating Assets:

A building delivers a flow of services, from construction or purchase time, until demolition or selling time. Usually, the service flow declines steadily, at least as the building gets old. Whether or not service flow declines, the building depreciates--because it approaches the end of its useful life. (It would depreciate even if its service flow remained constant, then suddenly ceased, like the one hoss shay). The amount of depreciation over the building's life just equals the cost of construction or purchase.

The building model applies at least roughly to any asset that yields a flow of services or income until replaced. Such assets include roads, machinery, reference books in a library, refrigerators, cars, clothing and "durables" in general. In addition, such assets include things that produce a continuous flow of physical output over their lives, such as fruit trees or power plants.

So the building model, like the tree model, also covers a broad range of production. In fact, most production can be treated as a combination of the tree and building models--such as a factory whose plant and equipment produce batches of goods for sale.

Note that the same asset may be appreciating or depreciating at different stages in its physical life. For example, a refrigerator appreciates on the manufacturer's assembly line; it then depreciates in the purchaser's kitchen.

So the building model says that richer people and bigger firms as a generalization carry a lower ratio of depreciating to non-depreciating assets. On the same quality land, they replace roads, buildings, machinery, orchards, etc. less frequently. On better land, they may replace more often.

A Comment on Mining:

The tree and building models do not quite fit one major form of economic activity: the mining of non-renewable resources. It seemed excessive to construct a separate mining model. However, mining poses some interesting problems.

It is obvious without a model that Large has a comparative advantage in owning better quality mines: better located, with higher grade ore, thicker seams--in general where extraction and transportation costs claim a lower share of output. It is also obvious that Large has a comparative advantage in holding mineral resources for appreciation before production begins, while Small has a comparative advantage in operating nearly-depleted mines.

But, does Large deplete a given mine slower or faster? Analogy with tree and building models suggests "slower". But the correct answer may depend on the characteristics of the mine.

Schematically, the cost of a mine has two components. First, there is the initial investment digging shafts or wells, building roads or laying pipes. Then there are extraction costs. The greater the initial investment--eg. the closer the shafts, the bigger the crushing plant--the greater the capacity of the mine. And the greater the capacity, the larger the flow of output for a given extractive cost, and the shorter the life of the mine.

Clearly, for a given initial capacity, Large extracts slower, due to higher labor costs. But does Large invest more or less in capacity? On the one hand, greater capacity saves future labor costs. But on the other hand, greater capacity shortens the life of the mine, possibly increasing labor costs over the life of the mine. The exact tradeoff may differ for different sorts of mines.

Note that, as for trees and buildings, a given mining company's production cycle may be shorter than the life of the mine. One company may hold a mine for appreciation from discovery to start of production. Another may mine it during its best years. And a third company may scratch out the remains. But, unlike tree or building owners, mine owners must necessarily buy new land at the end of each cycle of production.

Table 5.2
Sections 5.3 and 5.4

Notation for Tree and Building Models

w   wage of landowner, assumed higher for Large

r   discount rate of landowner

t   time

z   length of optimal cycle

b(t), b(z)   value of trees per acre as a function of time, value of trees at harvest time. Often abbreviated as simply "b".

$b >$ or $= 0$, $b' >$ or $= 0$, $b'' <$ or $= 0$, (usually).

f(t), f(z)   value of building service flow, as a function of time, value at demolition time. Often abbreviated as simply "f".

$f >$ or $= 0$, $f' <$ or $= 0$.

L_p planting labor per cycle per acre -- tree model

L_h harvest labor per cycle per acre -- tree model

$L_c = L_p + L_h$ total labor per cycle per acre -- tree model

L_b building labor per cycle per acre -- building model

L_m maintenance labor per acre -- building model

V   land value per acre

Y   gross income per acre = output per acre/ cycle length

L   labor per acre = L_c/z (tree model), $L_b/z + L_m$ (building model)

P   profit per acre = Y - wL

R   economic rent per acre, = rV

W   total value per acre, = P/r

Other symbols are defined when used in Tables 1, 3, and 4.

5.3 Derivation of the Tree Model[c]

Assumptions of the Tree Model:

a. The production function for trees is linear homogeneous in land and labor, so everything can be written in per acre terms.

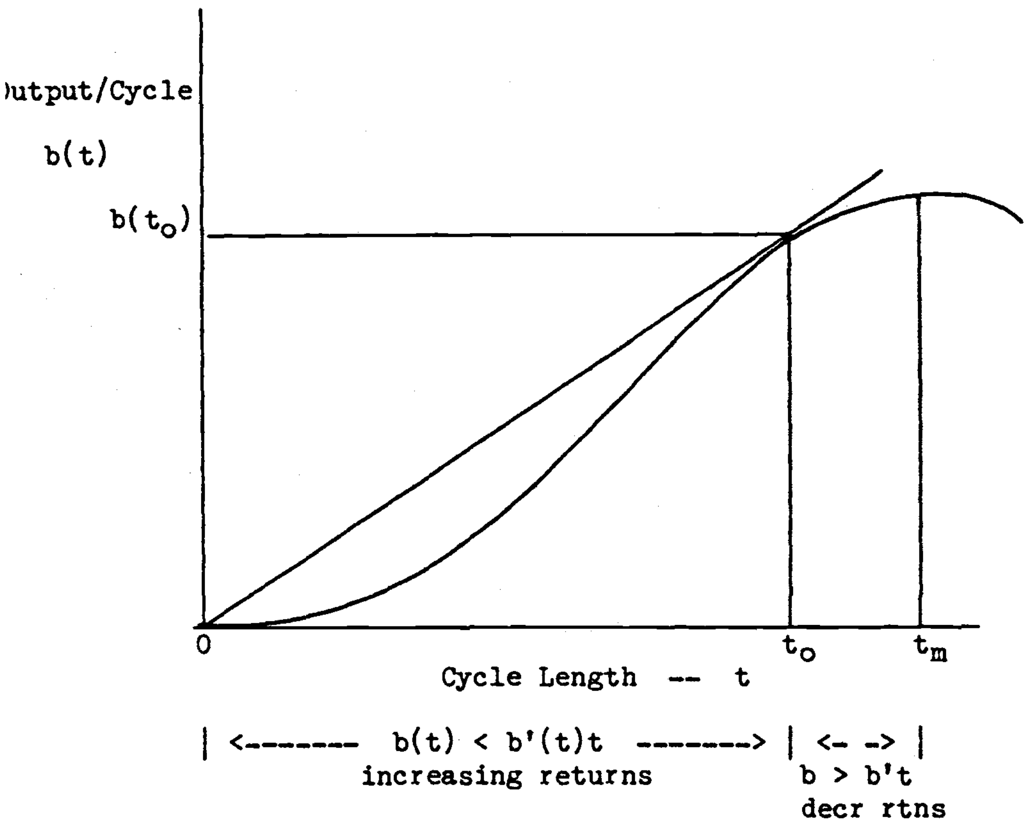
b. Tree production, b(t), depends solely on growing time. The only inputs are land, and planting and harvest labor per cycle, L_h and L_p, (the sum of which equals total labor per cycle, L_c.) Planting and harvest labor per cycle are fixed for any given piece of land.

c. Differences in land quality are modelled in two ways:
i. Planting and harvest labor requirements are lower on better land.
ii. Tree production is multiplied by a positive constant, k, which is higher on better land. Ie. tree production is written kb(t), with k higher on better land.

d. The trees are cut and planted in a staggered fashion; the same number of acres are cut and replanted each year. Nothing else changes from year to year either. Consequently, land value and all other functions remain the same each year. In other words, as in Chp. 4, tree farmers remain in a condition of dynamic equilibrium, keeping the same wealth from year to year.

e. The value of trees as a function of growing time, b(t), is S-shaped, as in Figure 5.1. The value increases at first slowly, then rapidly, then slowly again. The rate of growth, b'(t), goes to zero in a finite time (after which it may become negative--but the solution never lies in this region.)

The assumption of an S-shaped function means there is a region of increasing returns to time for small t, that is, where $b(t) < b'(t)t$, as shown in Figure 5.1. There is a constant returns point, followed by a



ig. 5.1: Tree model. Characteristics of typical production function. Output/cycle shows increasing returns up to t_o, where $b(t_o) = b'(t_o)t_o$, and decreasing returns beyond. t_m is the maximum possible length of a cycle. $b'(t_m) = 0$.

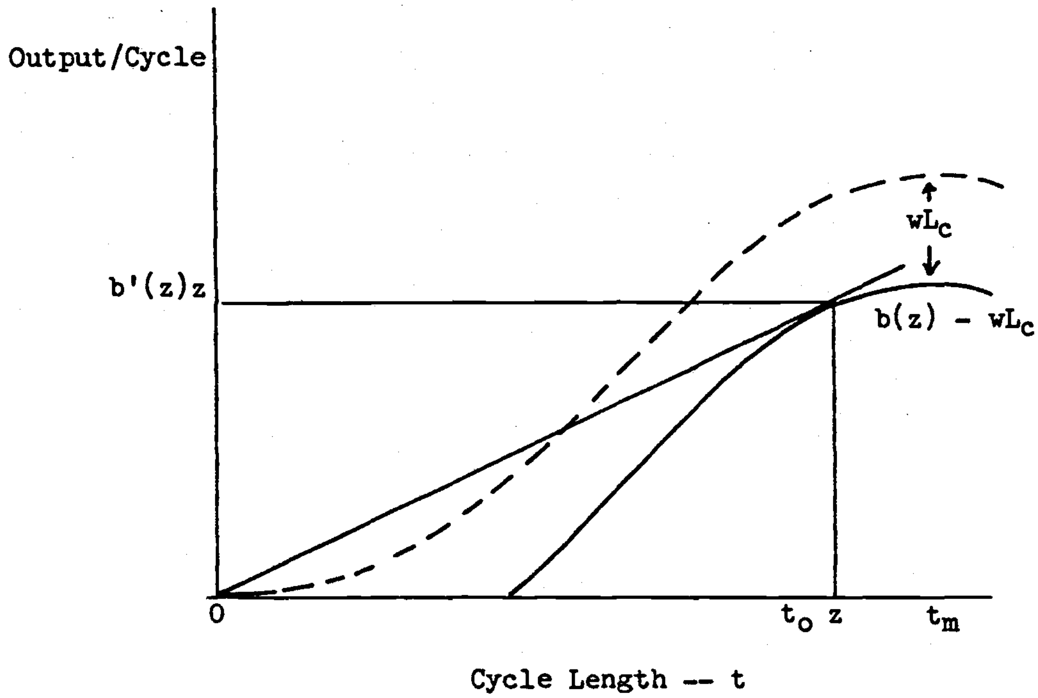


Fig. 5.2: Optimal cycle length, z. Profit maximizing solution: $b'(z)z = b(z) - wL_c$. Valid only for small rz. All solutions $z \geq t_o$.

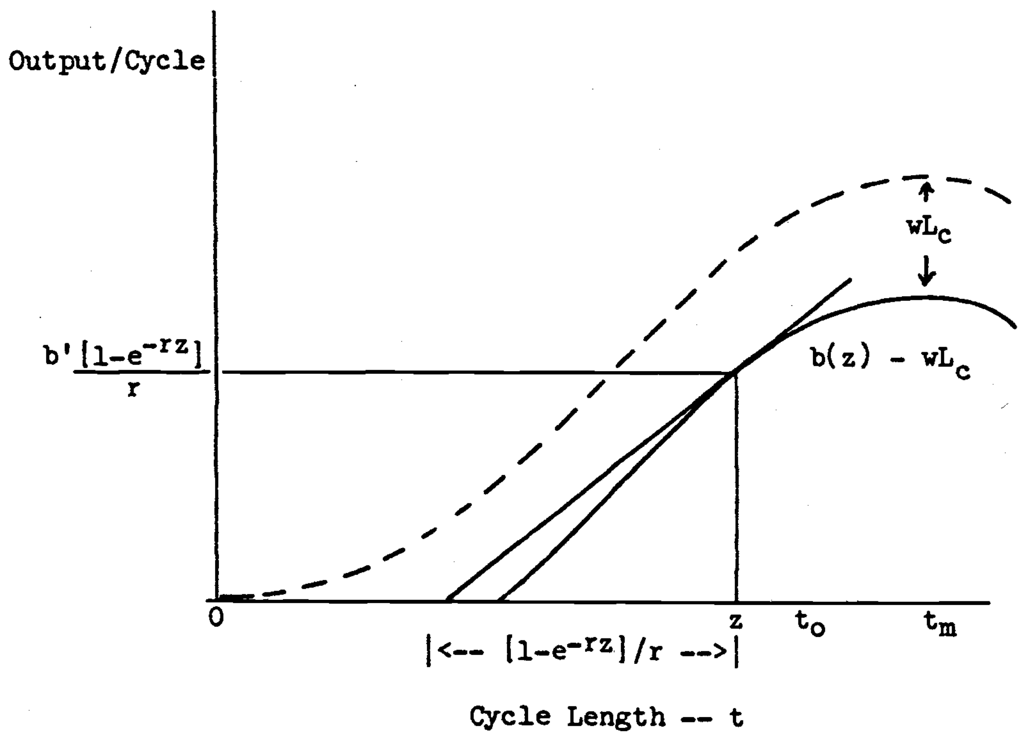


Fig. 5.3: Optimal cycle length, z. Present value maximizing solution: $b'(z)\frac{[1-e^{-rz}]}{r} = b(z) - wL_c$. Required for large rz.

region of decreasing returns to time. (If the bottom tail of the S is missing, so the function is everywhere concave, the constant returns point occurs at $t = 0$.) So it is assumed that the ratio $b'(t)t/b(t)$ falls steadily as t increases. (These restrictions on the production function eliminate some implausible cases which would require special treatment, without changing any results.)

<u>The Maximization Problem:</u>

The landowner determines his optimal cutting cycle, z, by maximizing the present value of his land.

This value equals the discounted present value of trees at harvest less harvest costs, minus planting costs, plus the discounted present value of land value at harvest:

$$(3.1) \qquad V = \frac{b(t) - wL_h}{e^{rt}} - wL_p + \frac{V}{e^{rt}}$$

Solving:

$$(3.2) \qquad V = \frac{b(t) - wL_h - e^{rt}wL_p}{e^{rt} - 1}$$

The same formula for per acre land value can be derived by recognizing that the harvest value net of harvest cost, $b(t) - wL_h$, must equal interest on planting costs, wL_p, plus interest on rent from time 0 to time t. Per acre rent R equals Vr -- the annual return on land value. Thus:

$$(3.3) \qquad b(t) - wL_h = wL_p e^{rt} + R\int_0^t e^{rx}dx$$

Which can be solved, substituting Vr for R, to obtain (3.2) again.

The landowner finds the optimal cycle, z, by taking the derivative of V with respect to t, and setting it equal to zero. He obtains from this the familiar Faustmann formula for the optimal cycle [eg. see Gaffney, 1960]:

(3.4) $$b'(z) = \frac{r[b(z) - w(L_p + L_h)]}{1 - e^{-rz}} = \frac{r[b(z) - wL_c]}{1 - e^{-rz}}$$

Given an explicit function b(t) for tree growth, this equation can be solved for the optimal cutting cycle, z.

At the optimal cycle, z, the land value becomes:

(3.5) $$V = \frac{b(z) - wL_h - e^{rz}wL_p}{e^{rz} - 1}$$

Notice that while planting labor cost, wL_p, enters into the land value formula with a factor of e^{rz}, it enters without that factor into the Faustmann formula. So, for simplicity, L_c, labor per cycle, can be substituted for $L_p + L_h$ in the Faustmann formula (3.4).

For very small rz, that is, $rz \ll 1$, e^{rz} and e^{-rz} reduce to 1, and $e^{rz} - 1$ and $1 - e^{-rz}$ reduce to rz. Then the Faustmann formula reduces to:

(3.6) $$b'(z) = \frac{b(z) - wL_c}{z}$$

Notice that r, the discount rate, no longer appears in the equation.

<u>Profit vs. Present Value Maximization:</u>

Profit per acre, P, equals gross income per acre ($Y = b(t)/t$), minus labor costs per acre ($L = L_c/t$):

(3.7) $$P = \frac{b(t) - wL_c}{t} = Y - wL$$

If the landowner maximizes profit he obtains a cycle length z*:

(3.8) $$b'(z^*) = \frac{b(z^*) - wL_c}{z^*}$$

z^*, the cycle obtained by maximizing profit, equals the optimal cycle, z, only if rz is small enough so that the short cycle Faustmann formula, (3.6), applies. Otherwise $z^* > z$; profit maximization gives too long a cycle. (This happens because $t > (1 - e^{-rt})/r$.)

So there is no "conflict" between "short-run" and "long-run" profit maximization. There is only present value maximization. However, profit maximization is a good approximation for present value maximization when production occurs on a short cycle, and/or discount rate is small.

(How short a cycle? Short enough for z to reasonably approximate $(1 - e^{-rz})/r$, or for $rz/(1 - e^{-rz})$ to approach one. If rz is .1, then $rz/(1 - e^{-rz})$ is about 1.05, a 5% error. So if r is 4% a year, z must be 2.5 years to get 5% error. If r is 10%, z must be 1 year.)

<u>A Graphic Solution:</u>

Figures 5.2 and 5.3 show a graphic solution to the present value maximization problem. First the curve of output per cycle, b(t) is dropped vertically by the amount of per cycle costs, wL_c. Then a straight line is drawn tangent to the curve with slope b'(t). When the base of the triangle formed with the horizontal axis is just $(1 - e^{-rt})/r$, then t = z, the solution. For small rz, the base of the triangle is just z, so the tangent line goes through the origin as in Figure 5.2.

<u>Range of Optimal Cycle, z, as Function of Labor Costs:</u>

As is apparent from inspection of the Faustmann formula (3.4), and (3.6) for very small rz, as well as the graphic solutions in Figures 5.2 and 5.3, the greater the labor costs per cycle, wL_c, the longer the cycle. And in fact, from the Faustmann formula:

$$(3.9) \qquad \frac{dz}{d(wL_c)} = \frac{r}{(b'r - b'')(1 - e^{-rz})} > 0$$

The range of the solution, z, runs from a minimum, z_{min}, for $wL_c = 0$, to a maximum, z_{max}, where land value V = 0. At the minimum, from (3.4):

$$(3.10) \qquad b'(z_{min}) = \frac{rb(z_{min})}{1 - e^{-rz_{min}}}$$

And if rz_{min} is small enough for profit-maximization to apply:

$$(3.11) \qquad b'(z_{min}) \dashrightarrow b(z_{min})/z_{min}$$

This is the point of constant returns to time. Graphically, as in Figure 5.1, it is the point where a straight line from the origin is tangent to the curve b(t).

Since $z > (1 - e^{-rz})/r$, then for larger rz, where (3.10) applies, the minimum solution must lie in the region of increasing returns to time.

The maximum value of z, z_{max}, must lie at the point where costs are so large as to make land value, V, equal to zero. Higher costs would make the land submarginal. So at maximum z, from (3.5):

$$(3.12) \qquad b(z_{max}) - wL_h - wL_p e^{rz_{max}} = 0$$

And for small rz, or for planting labor, $L_p = 0$, so that $L_h = L_c$:

$$(3.13) \qquad b(z_{max}) - wL_c = b'(z_{max}) = 0$$

So for small rz, or $L_p = 0$, the maximum z occurs where $b'(t) = 0$, that is, at the peak of the curve. Otherwise, as apparent from comparing (3.12) with (3.4), the Faustmann formula, maximum z occurs somewhere short of the peak, where $b'(t)$ is still > 0.

It is apparent from Figure 5.1 that the sharper the curve $b(t)$, the narrower the range of solutions.

Table 5.3, Sec. 5.6 shows the signs of derivatives for the 17 variables as summarized in Table 5.1. In a few cases a derivative has the same sign at either end of the range, but no obvious sign in the middle; in such cases it will be assumed "by continuity" that the sign remains the same over the whole range, from z_{min} to z_{max}. (The validity of this assumption could usually be demonstrated graphically, anyway.)

Comparative Behavior of Large and Small on the Same Quality Land:

If Large and Small grow trees on the same quality land, that land must have the same value to both, V_o. This fact constrains the Faustmann formula solutions such that:

$$(3.14) \qquad V_o = \frac{b(z) - wL_h - wL_p e^{rz}}{e^{rz} - 1} = \text{constant}$$

This constraint in turn makes it possible to compare the behavior of Large and Small on the same quality land, assuming that Large pays or imputes a higher wage, w.

As shown in Chp. 4, Large necessarily has a lower discount rate:

$$(3.15) \qquad \left.\frac{dr}{dw}\right|_{V_o} = -\frac{r(L_h + L_p e^{rz})}{b' z} < 0$$

And Large cuts trees on a longer cycle:

$$(3.16) \qquad \left.\frac{dz}{dw}\right|_{V_o} = \frac{L_h(1 + rz) + L_p e^{rz}}{(b'r - b'')z} > 0$$

These relationships can be used to find the effect of greater size, given the same quality land, on the other seventeen measures of economic behavior. The results appear in Table 5.3, second and third columns. The second column shows the effect of greater size as it would appear to an outside observer, who notices only that Large cuts on a longer cycle. The third column shows the effect as it appears to the forest owners themselves. A star (*) appears after those partial derivatives whose sign may vary, or is not obvious from inspection. Starred partials are treated further in Sec. 5.6, the notes to Table 5.3.

Comparative Advantage:

Sec. 5.6, notes to Table 5.3, shows that Large has a comparative advantage in owning better quality land--both land where the labor requirements L_p and/or L_h are lower, or the multiplicative constant, k, is higher. This follows from the fact that, on the same quality land, the rate at which land value rises with better quality is higher for Large than for Small. That is, although Large and Small may both own the same quality land at one point, better land is worth more to Large than to Small, while worse land is worth more to Small than to Large. Formally:

$$(3.17)\quad -\frac{d}{dw}\left(\frac{dV}{dL_p}\right)\bigg|_{V_0} > 0 \quad ; \quad -\frac{d}{dw}\left(\frac{dV}{dL_h}\right)\bigg|_{V_0} > 0$$

$$(3.18)\quad \frac{d}{dw}\left(\frac{dV}{dk}\right)\bigg|_{V_0} > 0$$

If Large has a comparative advantage in owning better quality land, then actual measures of differences between Large and Small, not holding land quality constant, may reflect effects of quality. If Large's land is sufficiently better than Small's, then the effect of better quality will dominate the effect of Large's higher wage. Columns 4 and 5 in Table 5.3 show the pure effects of better quality land on the seventeen economic measures. Derivatives with a star (*) are treated further in notes to Table 5.3.

5.4 Derivation of Building Model[c]

Assumptions of the Building Model:

a. The production function for buildings is linear homogeneous in land and labor, so everything can be written in *per acre* terms.

b. Service flow from a building, f(t), depends only on the age of the building, t. The only inputs are land, and building labor per cycle, L_b, and maintenance labor, L_m. Building labor per cycle, and maintenance labor are fixed for any given piece of land. (Note that maintenance labor is a constant flow through time, regardless of cycle, so that total labor per cycle equals $L_b + L_m z$, where z is cycle length.)

c. Differences in land quality are measured in two ways: i. Building and/or maintenance costs are lower on better land. ii. Service flow is multiplied by a positive constant, k, which is higher on better land. So service flow is written kf(t).

d. Buildings are built and demolished in a staggered fashion; the same area of land is razed and rebuilt each year.

e. Building service flow as a function of time, f(t), either declines steadily, or remains level for a while and then declines steadily. So $f'(t) < 0$ everywhere, or $f'(t) = 0$ at first, and then becomes < 0 as t increases. In either case f(t) reaches zero in a finite quantity of time. (A flow that rises at first, or one that decays

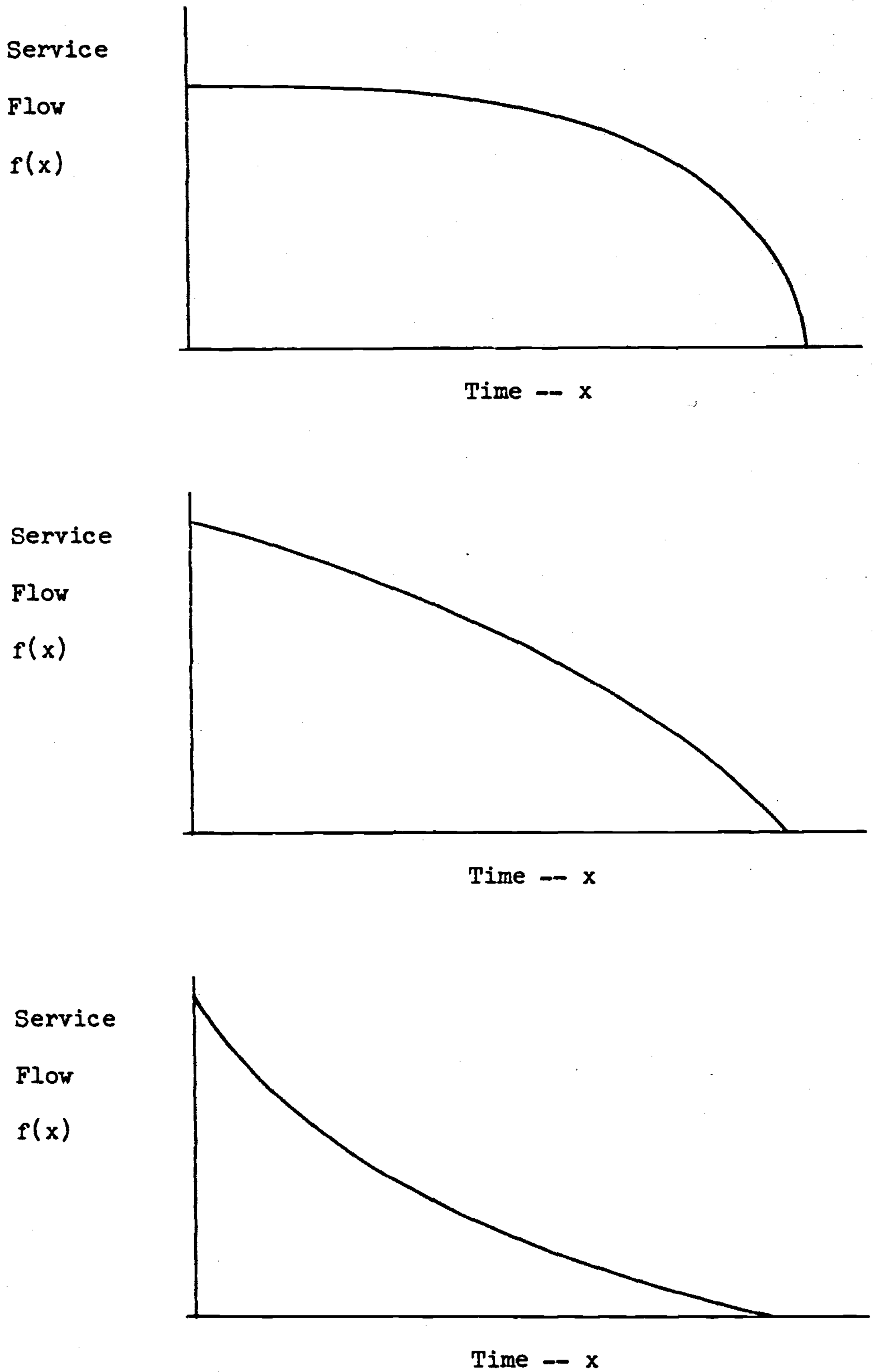


Fig. 5.4: Possible patterns of expected service flow from a durable asset like a building. Service flow must fall to zero in a finite time.

exponentially are excluded as they require special treatment, without altering any results.) Figure 5.4. shows three possible patterns of service flow.

The Maximization Problem:

The value of his land to a landowner, given his wage, w, and discount rate, r, -- equals the discounted present value of service flow less maintenance cost flow, less building cost, plus the discounted present value of bare land after the building is demolished. This comes out to:

$$(4.1) \qquad V = \frac{\int_0^t f(x)e^{-rx}dx - wL_b}{1 - e^{-rt}} - \frac{wL_m}{r}$$

The landowner maximizes the present value of his land to find the optimal cycle, z. He obtains the solution:

$$(4.2) \qquad f(z) = \frac{r\ [\int_0^z f(x)e^{-rx}dx - wL_b]}{1 - e^{-rz}}$$

Note first of all that this formula does not contain L_m, maintenance labor. For L_m is constant, regardless of the cycle.

Note second that the formula is almost, but not quite, the same as the Faustmann formula. It would be identical but for the e^{-rx} under the integral sign, with b(z) equivalent to $\int_0^z f(x)dx$, output per cycle, and b'(z) equivalent to f(z), rate of increase in output per cycle.

The formula (4.2) for optimal building replacement cycle can be rewritten in a more familiar form:

$$(4.3) \qquad f(z) - wL_m = Vr \quad (= R)$$

The optimal time to replace the building occurs when service flow less maintenance cost just covers interest on land value, which equals the economic rent. This is a "pseudo" solution, as the actual solution (4.2) does not depend on L_m, and land value V is internally determined, not given.

For very small rz, that is, rz << 1, e^{-rz} and e^{-rx} reduce to 1, and $1 - e^{-rz}$ reduces to rz. Then the optimal cycle formula, (4.2), reduces to:

$$(4.4) \qquad f(z) = \frac{\int_0^z f(x)dx - wL_b}{z}$$

Like the Faustmann formula for small rz, (4.6), this formula does not contain r, the discount rate. And it is in fact identical to the Faustmann formula, equating b(z) to $\int_0^z f(x)dx$, output per cycle.

<u>Profit Maximization:</u>

Profit per acre, P, equals gross income per acre, $Y = \frac{1}{z}\int_0^z f(x)dx$; minus labor costs per acre, $wL = wL_b/z + wL_m$:

$$(4.5) \qquad P = \frac{\int_0^z f(x)dx - wL_b}{z} - wL_m = Y - wL$$

If the landowner maximizes profit, he obtains a cycle length z*:

$$(4.6) \qquad f(z^*) = \frac{\int_0^{z^*} f(x)dx - wL_b}{z^*}$$

It is apparent from equation (4.4), (the formula that applies for small rz), that if rz is small, then z* = z, the optimal cycle. Otherwise profit maximization yields a cycle length, z*, that is <u>too</u>

short, --as is obvious from the fact that $f(z^*) > f(z)$, which means $z^* < z$, since $f(t)$ declines as t increases. So while profit maximization gives too long a cycle for appreciating assets, it gives too short a cycle for depreciating assets!

A Graphic Solution:

Figure 5.5 shows a graphic solution to the maximization problem in terms of service flow. This is a representation of the "pseudo" solution in equation (4.3). Figures 5.6, 5.7, and 5.8 show a correct, but less revealing, graphic solution in terms of output per cycle, analogous to the graphic solution for the tree model. Figure 5.6 shows the curve of discounted output per cycle, $\int_0^t f(x)e^{-rx}dx$. The slope of this curve at any point is $f(t)e^{-rt}$. In Figures 5.7 and 5.8, the curve is shifted downwards by the amount of building costs, wL_b. In Figure 5.7, the profit-maximizing solution for small rz, the solution lies at the point where a straight line from the origin is tangent to the curve. In Figure 5.8, the present-value maximizing solution, the solution lies at the point where a straight line tangent to the curve forms a triangle with the horizontal axis, with base $(e^{rz} - 1)/r$, which is $> z$. This solution represents equation (4.2), with both sides multiplied by e^{-rz}.

Range of Optimal Cycle, z, as Function of Labor Costs:

As is apparent from inspection of (4.2) and (4.4), as well as the graphic solutions in Figures 5.7 and 5.8, the greater the building labor costs per cycle, wL_b, the longer the optimal cycle, z. And in fact from the solution (4.2):

$$(4.7) \qquad \frac{dz}{d(wL_b)} = - \frac{r}{f'(z)(1 - e^{-rz})} > 0$$

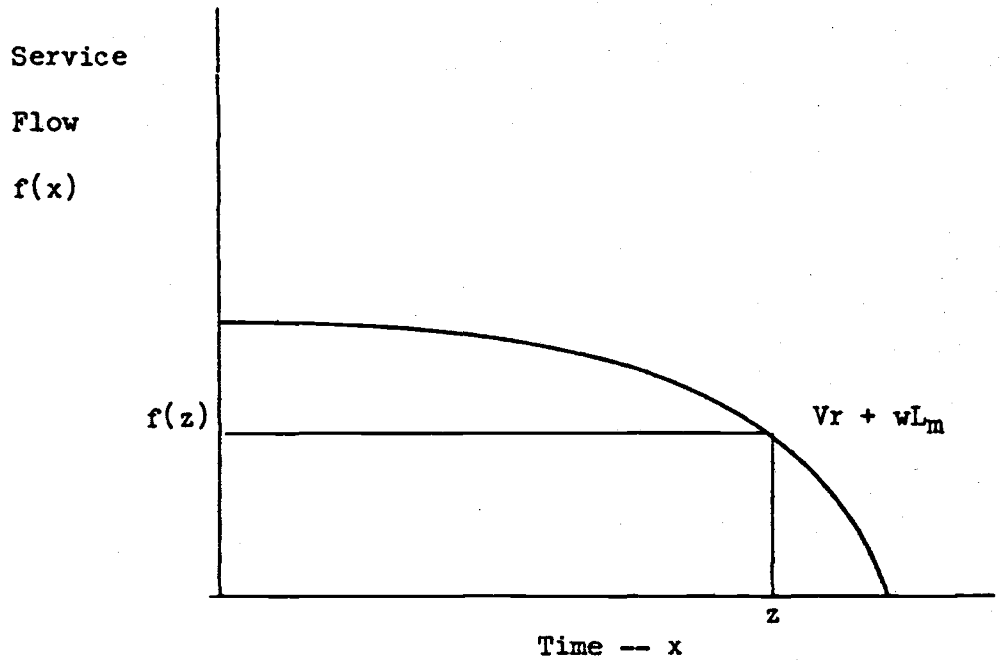


Fig. 5.5: Pseudo solution to maximization problem in terms of service flow: $f(z) = Vr + wL_m$. Service flow at optimal life z equals interest on land value plus cost of maintenance labor.

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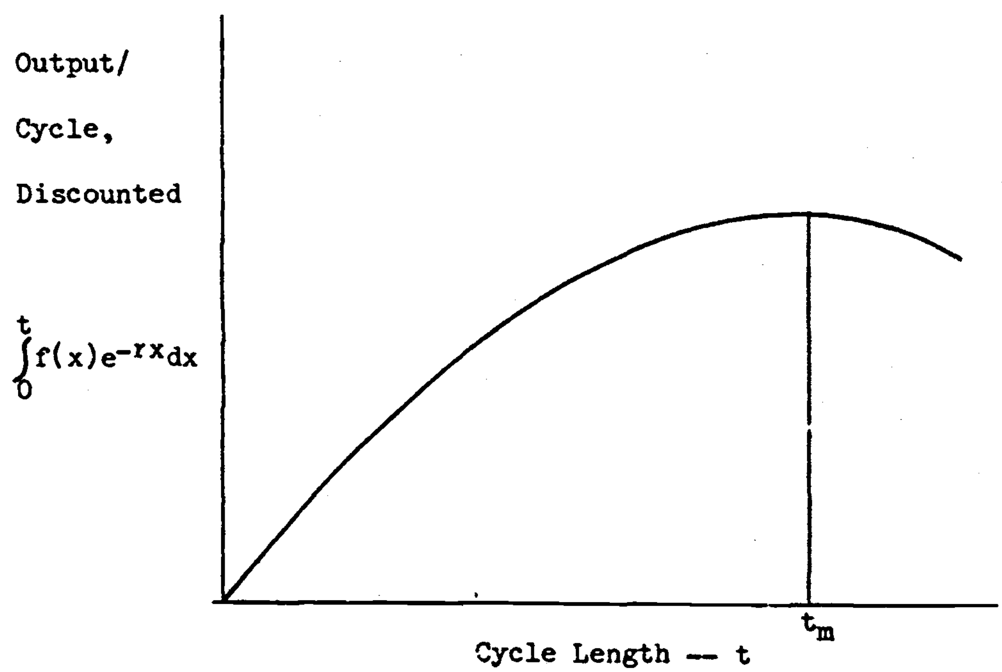

Fig. 5.6: Output/cycle, discounted, $\int_0^t f(x)e^{-rx}dx$ as a function of cycle length, t. t_m = maximum possible cycle length, at $f(x)e^{-rx} = 0$.

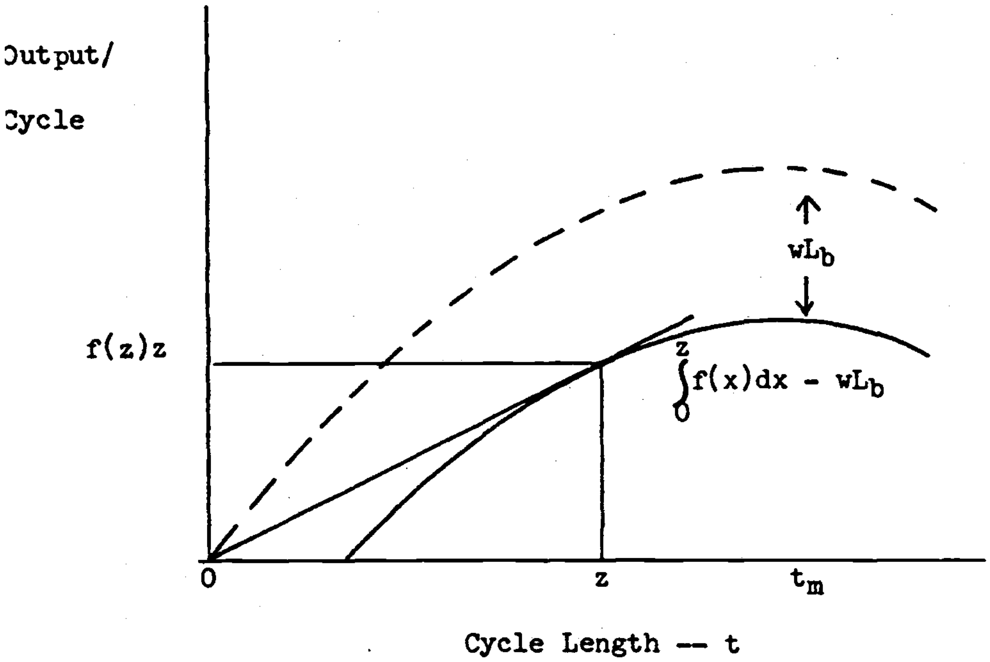

Fig. 5.8: Optimal cycle length, z. Profit-maximizing solution, $f(z)z = \int_0^z f(x)dx - wL_b$. Valid only for small rz.

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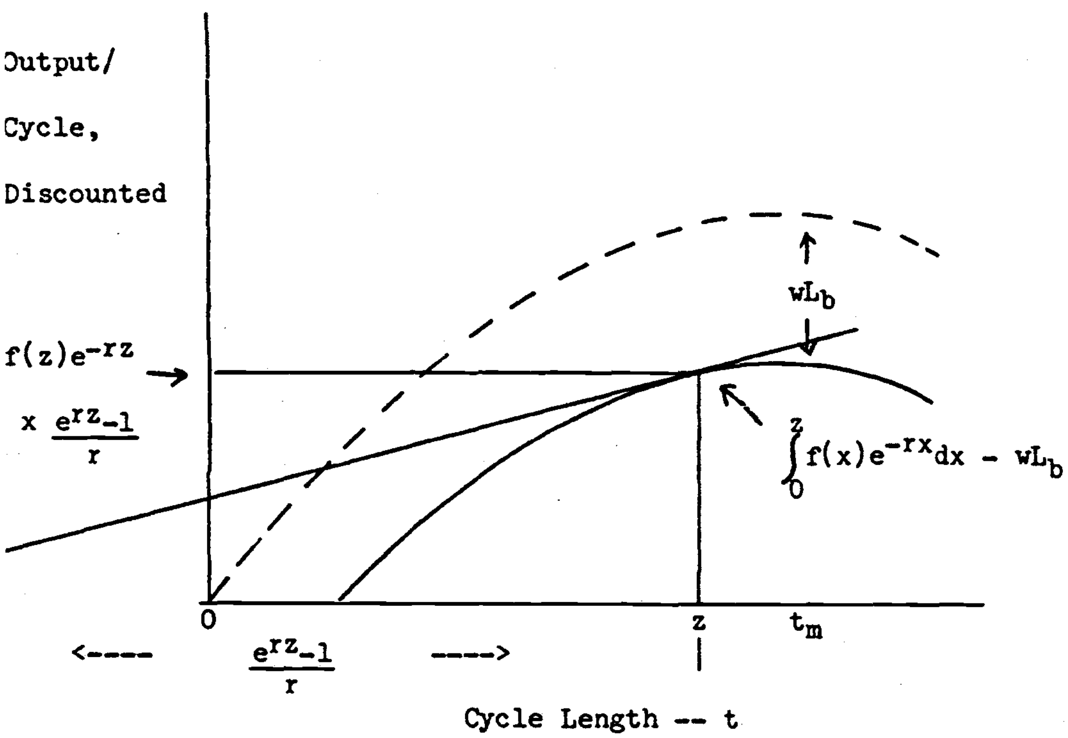

Fig. 5.8: Optimal cycle length, z. Present value maximizing solution,

$$f(z)e^{-rz}\,\frac{(e^{rz}-1)}{r} = \int_0^z f(x)e^{-rx}dx - wL_b.$$

The range of the solution, z, runs from a minimum, z_{min}, for $wL_b = 0$; to a maximum, z_{max}, where land value V = 0. At the minimum, from (4.2):

$$(4.8) \qquad f(z_{min}) = \frac{r\int_0^{z_{min}} f(x)e^{-rx}dx}{1 - e^{-rz_{min}}}$$

If f(t) declines steadily, the only possible solution to this is $z_{min} = 0$. If f(t) remains level before declining, any value in that plateau is a solution; but assume there is only one "true" solution z_{min}, at the point at which f(t) starts to decline.

The maximum solution, z_{max}, occurs where V = 0. From (4.3), this is the point where $f(z_{max}) = wL_m$, that is, where service flow just equals maintenance costs. So while maintenance labor, L_m, does not figure in the optimal solution, it does set the far end of the range of optimal solutions. The larger wL_m, the shorter the range.

Comparative Behavior of Large and Small on the Same Quality Land:

If Large and Small construct buildings on the same quality land, that land must have the same value to both, V_o. As in the tree model, this constrains the optimal solution formula (4.2) and the "pseudo" solution (4.3) so that:

$$(4.9a) \qquad V_o = \frac{\int_0^{z} f(x)e^{-rx}dx - wL_b}{1 - e^{-rz}} - \frac{wL_m}{r} = \text{constant}$$

$$(4.9b) \qquad V_o = \frac{f(z) - wL_m}{r} = \text{constant}$$

As in the tree model, this constraint makes it possible to compare the behavior of Large and Small on the same quality land, assuming that

Large pays or imputes a higher wage, w.

Large of course has a lower discount rate:

$$(4.10) \quad \left.\frac{dr}{dw}\right|_{V_o} = \frac{\frac{L_m}{r} + \frac{L_b}{(1 - e^{-rz})}}{\frac{dV}{dr}} < 0$$

-- since dV/dr < 0, as shown in notes to Table 4.

And as in the tree model, Large rebuilds on a longer cycle, z. That is:

$$(4.11) \quad \left.\frac{dz}{dw}\right|_{V_o} = \frac{\frac{L_b V_o}{1 - e^{-rz}} + \frac{L_m}{r}\left(V_o + r\frac{dV}{dr}\right)}{f'\frac{dV}{dr}} \geq 0 \qquad (= 0 \text{ for } L_b = 0)$$

-- as is shown in notes to Table 4.

As in the tree model, these relationships can be used to find the effect of greater size, given the same quality land. The results appear in the second and third columns of Table 5.4. The second column shows the effects as they look to an observer who notices only that Large replaces buildings on a longer cycle. The third column shows the effects as they look to the landowners themselves. Starred partials (*) are treated further in notes to Table 4.

<u>Comparative Advantage</u>:

As with the tree model, Sec. 5.7, notes to Table 4, shows that Large has a comparative advantage in owning better quality land. This is land where the labor requirements L_b and or L_m are lower, or the factor multiplying output, k, is higher. As before, Large's comparative advantage follows from the fact that, on the same quality land, the rate at which land value rises with better quality is higher for Large than for Small.

Formally:

(4.12) $$-\frac{d}{dw}\left(\frac{dV}{dl_b}\right)\Big|_{V_0} > 0 \; ; \quad -\frac{d}{dw}\left(\frac{dV}{dl_m}\right)\Big|_{V_0} > 0$$

(4.13) $$\frac{d}{dw}\left(\frac{dV}{dk}\right)\Big|_{V_0} > 0$$

Comparative advantage of Large on better land also follows from the lower intrinsic labor intensity of production on better land, as evident in the lower labor share.

If Large has a comparative advantage in owning better quality land, then actual measures of differences between Large and Small, not holding land quality constant, may reflect effects of quality. If land quality increases substantially with size, then the effect of better quality may dominate the effect of size. Columns 4 and 5 in Table 5.4 show the pure effects of better quality land on the seventeen economic measures. Derivatives with a star (*) are treated further in notes to Table 5.4.

5.5 Comparative Advantage and Cycle Length

All else being equal, longer cycle activities have a higher average product of labor. This implies that Large has a comparative advantage in longer cycle activities.

Example from the tree model:

Suppose there are two production functions, b(t) and b*(t), where t is time. Labor requirements per cycle are L_p, L_h, and L_p*, L_h* respectively.

For a given wage, w, and discount rate r, the optimal cycle lengths are z and z*, respectively. So the average products of labor at optimal cycle lengths are:

$$(5.3) \qquad AP = \frac{b(z)}{L_p + L_h} \quad ; \qquad AP^* = \frac{b^*(z^*)}{L_p^* + L_h^*}$$

And the land values are:

$$(5.4) \qquad V = \frac{b(z) - wL_h - wL_p e^{-rz}}{e^{rz} - 1}$$

$$V^* = \frac{b^*(z^*) - wL_h^* - wL_p^* e^{-rz^*}}{e^{rz^*} - 1}$$

Now assume that $z^* > z$, but land values are the same, ie. $V = V^*$.

For $z^* > z$, the denominator of V^* is greater than the denominator of V, ie. $e^{rz^*} - 1 > e^{rz} - 1$. Therefore the numerators:

$$(5.5) \qquad b^*(z^*) - wL_h^* - wL_p^* e^{rz^*} > b(z) - wL_h - wL_p e^{rz}$$

From this inequality it follows that $b^*(z^*) > b(z)$, and/or $L_h^* < L_h$, and/or $L_p^* < L_p$. And from these inequalities it follows that:

$$(5.6) \qquad AP^* > AP$$

So, all else being equal, the average product of labor is higher for longer cycle activities.

Table 5.3 1
TREE MODEL

	Economic Measure	Effect of Greatr Wealth on Same Qualty Land: To Observer: d/dz	Effect of Greatr Wealth on Same Qualty Land: To Owner: d/dw V_o	Effect of Better Quality Land: Lower Labor: $-d/dL_c$	Effect of Better Quality Land: Higher Prod: d/dk
1	Cycle length: z	$1 > 0$	$\frac{L_h(1+rz) + L_p e^{rz}}{(b'r - b'')z} > 0$ *	$\frac{-wr}{(b'r-b'')(1-e^{-rz})} < 0$ *	$\frac{-rwL_c}{k^2(b'r-b'')(1-e^{-rz})} < 0$ *
2	Cycle x discnt: rz	$r > 0$	< 0, mostly*	$-r \frac{dz}{dL_c} < 0$	$r \frac{dz}{dk} < 0$
3	Output/cycle: $OP = b(z)$	$b' > 0$	$b' \frac{dz}{dw} > 0$	$-b' \frac{dz}{dl_c} < 0$	$b + kb' \frac{dz}{dk} > 0$ †
4	Gross income/acre: $Y = b(z)/z$	$\frac{-(b - b'z)}{z^2}$ > 0 i.r.†† < 0 d.r.	$\frac{dY}{dz}\frac{dz}{dw}$ > 0 i.r. < 0 d.r.	$-\frac{dY}{dz}\frac{dz}{dL_c}$ < 0 i.r. > 0 d.r.	$\frac{1}{z}\left(\frac{dOP}{dk} - \frac{OP}{z}\frac{dz}{dk}\right) > 0$
5	Labor/acre: $L = \frac{L_c}{z}$	$-\frac{L_c}{z^2} < 0$	$\frac{dL}{dz}\frac{dz}{dw} < 0$	$\frac{-1}{z}\left(1 - \frac{L_c}{z}\frac{dz}{dL_c}\right) < 0$, mstly**	$-\frac{L_c}{z^2}\frac{dz}{dk} > 0$
6	Labor cost/acre: $LC = wL$	$-\frac{wL_c}{z^2} < 0$	$\frac{L_c}{z}\left(1 - \frac{w}{z}\frac{dz}{dw}\right) > 0$, mostly**	$\frac{-w}{z}\left(1 - \frac{L_c}{z}\frac{dz}{dL_c}\right) < 0$, mstly**	$w \frac{dL}{dk} > 0$
7	Rent/acre: $R = rV$	0	$V_o\frac{dr}{dw} = -V_o\frac{r(L_h+L_p e^{rz})}{b'z} < 0$	$\frac{-dR}{dL_h} = \frac{rw}{e^{rz}-1}$, $\frac{-dR}{dL_p} = \frac{rw}{1-e^{-rz}} > 0$	$r \frac{dV}{dk} > 0$
8	Profit/acre: $P = Y - wL$	> 0 to z^*_{ob} thn < 0*	< 0 * $\frac{dP}{dz} = \frac{b'[rz-(1-e^{-rz})]}{z^2 r} > 0$	$\frac{w}{z} - \frac{dP}{dz}\frac{dz}{dL_c} >$ $\frac{w}{z}\frac{[e^{rz}-(1+rz)]}{rz(e^{rz}-1)} > 0$	$\frac{b}{z} + \frac{kb'[rz-(1-e^{-rz})]}{rz^2}\frac{dz}{dk}$ > 0 †
9	Av prod labor: $AP = \frac{b(z)}{L_c}$	$\frac{b'}{L_c} > 0$	$\frac{b'}{L_c}\frac{dz}{dw} > 0$	$\frac{b}{L_c^2}\left[1 - \frac{b'L_c wr}{b(b'r-b'')(1-e^{-rz})}\right]$ > 0 †	$\frac{1}{L_c}\frac{dOP}{dk} > 0$

Table 5.3.2

	Economic Measure	Effect of Greatr Wealth on Same Qualty Land		Effect of Better Quality Land	
		To Observer: d/dz	To Owner: $d/dw \mid V_o$	Lower Labor: $-d/L_c$	Higher Prod: d/dk
10	Labr shre of outpt: $LS = \frac{wL_c}{b} = \frac{w}{AP}$	$-\frac{wL_c b'}{b^2} < 0$	$\frac{L_c}{b}\left[1 - \frac{b'w}{b}\frac{dz}{dw}\right] > 0$†	$\frac{w}{(AP)^2}\frac{dAP}{dL_c} < 0$	$-\frac{w}{(AP)^2}\frac{dAP}{dk} < 0$
11	Rent share: $RS = \frac{R}{Y}$	$-\frac{R}{Y^2}\frac{dY}{dz}$ < 0 i.r. > 0 d.r.	$RS = V_o\frac{rz}{b}$; $\frac{dRS}{dw} < 0$*	$-\frac{dRS}{dL_c} > 0$ *	$\frac{dRS}{dk} > 0$ *
12	Profit share: $PS = 1 - LS$	$-\frac{dLS}{dz} > 0$	$-\frac{dLS}{dw} < 0$	$- -\frac{dLS}{dL_c} > 0$	$-\frac{dLS}{dk} > 0$
13	Land value: $V = \frac{b - wL_h - wL_p e^{rz}}{e^{rz} - 1}$	0	0	$-\frac{dV}{dL_h} = \frac{w}{e^{rz}-1} > 0$ $-\frac{dV}{dL_p} = \frac{w}{1-e^{-rz}} > 0$	$\frac{b}{e^{rz}-1} > 0$
14	Total value: $W = \frac{P}{r}$	$\frac{1}{r}\frac{dP}{dz}$ or $\frac{dW}{dz} > 0$ *	> 0, $L_p \gg L_h$ * < 0, $L_h \gg L_p$	$-\frac{1}{r}\frac{dP}{dL_c} > 0$	$\frac{1}{r}\frac{dP}{dk} > 0$
15	Improvement value: $IM = W - V$	$\frac{dIM}{dz} = \frac{dW}{dz}$	$\frac{dIM}{dw} = \frac{dW}{dw}$	$-\frac{dIM}{dL_p} < 0$ * $-\frac{dIM}{dL_h}$ < 0 sm rz > 0 lg rz	$\frac{dIM}{dk} = \frac{1}{krz}\left[kb\left(1 - \frac{rz}{e^{rz}-1}\right) - \frac{wL_c b' r}{b'r - b''}\left(\frac{1}{1-e^{-rz}} - \frac{1}{rz}\right)\right] > 0$†
16	Ratio impr to land value: $RT = IM/V$	$\frac{dW}{dz}$	$\frac{dW}{dw}$	$-\frac{dRT}{dL_c} = -\frac{d}{dL_c}\frac{W}{V} < 0$*†	$\frac{dRT}{dk} = \frac{d}{dk}\frac{W}{V} < 0$ *†
17	Capital turnover: $TN = \frac{Y}{W} = \frac{r}{PS}$	$-\frac{r}{(PS)^2}\frac{dPS}{dz} < 0$ *	$\frac{dTN}{dw}$ < 0, $L_p \gg L_h$, or very sm z > 0, $L_h \gg L_p$	$\frac{r}{(PS)^2}\frac{dPS}{dL_c} < 0$	$-\frac{r}{(PS)^2}\frac{dPS}{dk} < 0$

Footnotes to Table 5.3

* Discussed further in notes to Table 5.3 below.

**Sign depends on elasticity of z wrt. w or L_c, $\frac{w}{z}\frac{dz}{dw}$; $\frac{L_c}{z}\frac{dz}{dL_c}$. These elasticities will be < 1 for small w or L_c, and in general for most functions. The elasticity for large w or L_c, near the peak of the curve b(t), depends on how rapidly b(t) approaches its maximum, that is, on the magnitude of $-b''$. The larger $-b''$, the faster the approach to the maximum, and the smaller the elasticity.

† Sign determined by assumption that sign for min w or L_c, where w or $L_c = 0$; and for max w or L_c, where $b' = 0$; holds between the min and max.

†† Increasing returns to time (i.r.), as described above, may occur for small w and large rz.

*† Ratio of improvements to land value falls with better land, because cycle length z falls; the shorter the cycle, the closer total value to land value.

5.6 Notes to Table 5.3D

Effect of Greater Wealth on Same Quality Land, To Observer (Col 2).

It is assumed that an outside observer knows the production function b(t) of a given quality of land, and the planting and harvest labor requirements, L_p and L_h. But if he sees landowners of different wealth on the same quality land, he is not able to measure their internal wages and discount rates. So he imputes to them a single wage and discount rate, his own or "market".

However, an observer is able to measure different landowners' cycle length, z, or, what amounts to the same thing, their gross income per acre, b(z)/z. Note that the cycle lengths, z, while optimal for different landowners, are not optimal for the observer, unless they equal the optimal cycle length, z_{ob}, for his particular wage and discount rate.

For items 1. through 13., the observed differences between landowners are simply the direct consequence of differences in observed cycle length, z. However, for items 14. through 17., total value, improvement value, ratio of improvement to land value, and capital turnover, there is a problem: Does the observer assume that the same cycle length, z, will persist into the future, -- or does he assume that in the future the (to him) optimal cycle length, z_{ob}, will apply? (This is the problem any appraiser faces: Does he assume that his client will continue to manage as in the past, and value the client's assets accordingly, --or does he assume the client's management will "improve" in the future?) It seems more plausible that the observer will assume that z_{ob} will apply in the future. Results under either assumption are discussed below.

8. Profit per acre: $P = Y - wL$

For the observer, there is a value of z, z^*_{ob}, where profit is a maximum. For very small rz, $z^*_{ob} = z_{ob}$; for larger rz, $z^*_{ob} > z_{ob}$. (This is discussed under "Derivation of the Tree Model, section g.) So to the observer, the profit per acre of landowners on the same quality land rises with the landowners' wealth until the landowners' cycle length reaches z^*_{ob}; then it falls again.

14. Total Value per acre: W

There are two possible assumptions about how the observer measures total value.

a. The observer may simply assume the landowners will use the same cycle length in the future. That means he assumes profit will continue the same in the future, and total value just equals discounted present value of future profit: $W = P/r$. Therefore, total value rises and falls with profit, ie:

$$\frac{dW}{dz} = \frac{1}{r}\frac{dP}{dz}$$

But the results of this assumption are implausible. For it is "common sense" that the older on average a stock of timber, the more valuable it is. (Recall that the maximum possible cycle length is the one for which b(z) is a maximum, and $b' = 0$, --so it is not possible to have a cycle length so long that the trees are deteriorating when cut.)

b. The observer may assume that the "correct" cycle length, z_{ob}, will apply in the future. This assumption will give him the common sense result that an older stand of timber is more valuable than a younger one.

Formally, imagine that the land is divided into z cells, numbered $s = 1 \ldots z$. There are trees on each cell, of age s.

The total value of each cell must be:

$$W_s = (b(z_{ob}) - wL_h + V)e^{-r(z_{ob} - s)} \quad \text{for } s \leq z_{ob}$$

$$W_s = b(s) - wL_h + V \quad \text{if } s > z_{ob}$$

That is, the observer assumes the trees in each cell will be allowed to grow to age z_{ob}. If they already exceed z_{ob}, they will be cut immediately.

The observer perceives an average total value $W_<$ for $z \leq z_{ob}$, and $W_>$ for $z \geq z_{ob}$:

$$W_< = \frac{1}{z}\int_0^z W_s \, ds \quad \text{for } z \leq z_{ob}$$

$$= e^{-rz_{ob}} [b(z_{ob}) - wL_h + V] \frac{(e^{rz} - 1)}{rz}$$

$W_<$ obviously increases as z increases.

and:

$$W_> = \frac{1}{z}\left[\int_0^{z_{ob}} W_s \, ds \quad s \leq z_{ob} \right. \qquad \text{for } z \geq z_{ob}$$

$$\left. + \int_{z_{ob}}^{z} W_s \, ds \right] \quad s > z_{ob}$$

$$W_{>} = \frac{1}{z} \left[(b(z_{ob}) - wL_h) \frac{(1-e^{-rz_{ob}})}{r} + \int_{z_{ob}}^{z} (b(s) - wL_h)ds \right]$$

$$+ V \left[1 - \frac{1}{z} \frac{(rz_{ob} - (1-e^{-rz_{ob}}))}{r} \right]$$

The second line of $W_{>}$ obviously increases as z increases. The first can easily be shown to increase using the fact that the integral is $\geq$ $(b(z) - wL_h)(z - z_o b)$.

So the observer always perceives total value to increase with wealth. Since land value is a constant, he also perceives improvement value to increase, and the ratio of improvement to land value.

17. Capital Turnover: TN = Y/W = r/PS

The observer measures capital turnover differently, depending on which assumption he makes about total value. In other words, to the observer, Y/W $\neq$ r/PS. But in either case, he perceives capital turnover to fall as wealth increases.

Effect of Greater Wealth on Same Land, To Owner (Column 3):

Since it is assumed that Large has a higher wage, $\left.\frac{d}{dw}\right|_{V_o}$ ($\frac{d}{dw}$ for short) gives the direction of change with increased wealth.

2. Cycle Length x Discount Rate: rz

$$\frac{drz}{dw} = r\frac{dz}{dw} + z\frac{dr}{dw}$$

$$= \frac{r \left[L_h(b'+b''z) + L_p e^{rz}(b'(1-rz) + b''z) \right]}{(b'r - b'')\, b'z}$$

This expression must be < 0 towards maximum z, where b' gets very small, so that (negative) b" dominates.

It will also be < 0 near minimum z (where w = 0), if the point where b' = -b"t occurs before minimum z -- which depends on the shape of the curve b(t).

So rz may be falling everywhere, or it may rise awhile and then fall.

If a maximum rz exists, it must equal:

$$(rz)_{max} = 1 + \frac{L_h}{L_p e^{rz}} + \frac{b''z}{b'} [1 + \frac{L_h}{L_p e^{rz}}]$$

$$= 1 + \frac{b''z}{b'} < 1 \qquad \text{for} \quad L_h = 0$$

<u>8. Profit per acre:</u> P = Y - wL

Profit can be written two ways:

$$P = \frac{b - wL_e}{z} \underset{(\text{from } (3.4))}{=} \frac{b'(1 - e^{-rz})}{rz} > 0$$

It is obvious from the both versions that P must fall as z increases, at least for large z where b' approaches 0. The second version will obviously fall where b' is falling and rz is increasing. By writing out the derivatives it can be shown that P does in fact fall everywhere.

dP/dz in Table 3 is obtained by taking dP/dz and then substituting the second version of P above:

$$\frac{dP}{dz} = \frac{1}{z} [b' - \frac{P}{z}] = \frac{b'[rz - (1-e^{-rz})]}{z^2 r}$$

Note that it is not possible to make the substitution for the observer, above, since the Faustmann formula (3.4) does not apply for $z \neq z_{ob}$.

<u>11. Rent Share:</u> $RS = \frac{R}{Y} = V_o\frac{rz}{b}$

Rent share obviously falls when rz falls and b increases as w (and wealth) increases.

What about the region for small w where rz may increase? Rent share can be rewritten as the product of two terms:

$$RS = \frac{rz}{e^{rz} - 1} \cdot \frac{b - wL_h - wL_p e^{rz}}{b}$$

Both of these terms fall if rz increases as w increases.

So rent share falls as w (wealth) increases on the same land -- as it should, since it is identical to profit share for small rz.

<u>14. Total Value per acre:</u> $W = \frac{P}{r} = [b(z) - wL_h + V]\,\frac{(1-e^{-rz})}{rz}$

It is apparent from the second expression for W, that W can equal V only if z = 0 and b(0) = 0 when w = 0, (though b(z)/z > 0 at z = 0).

For large rz:

$$\frac{dW}{dw} = \frac{L_h}{(rz)^2}\,\frac{[b' + b''z\,]}{(b'r-b'')z}\,[rz - (1 - e^{-rz})]$$

$$+ \frac{L_p}{(rz)^2}\,\left[\frac{e^{rz}}{1 - \frac{b''}{b'r}}\,\frac{rz - (1 - e^{-rz})}{rz} + e^{rz}(1 - e^{-rz}) - rz\right]$$

The L_p term is always > 0.

The L_h term is > 0 for b' > - b"z, which may be the case for small w, specially if z = 0 for w = 0 (a concave production function.) It is < 0 for ' < -b"z, which is necessarily true for larger w, as b' becomes very small.

For $L_p = 0$:

$$W = V_o\,\frac{e^{rz} - 1}{rz}$$

which obviously falls when rz falls.

15. Improvement Value per acre: $IM = W - V_o$

$$IM = (b - wL_h)\left[\frac{1}{rz} - \frac{1}{e^{rz} - 1} \right] + wL_p \left[\frac{1}{1 - e^{-rz}} - \frac{1}{rz} \right]$$

$$= \frac{b - wL_h + wL_p}{2} \quad \text{for small } rz$$

Notice that L_h and L_p enter into improvement value with opposite signs. Improvement value will = 0 only at w = 0 and only in the case of a strictly concave production function such that b(0) = 0. (Gross income $Y = b(z)/z$ will still be > 0.)

Since $IM = W - V_o$, $dIM/dw|V_o = dW/dw|V_o$.

16. Capital Turnover: $TN = \frac{Y}{W} = \frac{r}{PS} = \frac{rb}{b - wL_c}$

For large rz:

$$\frac{dTN}{dw} = \frac{b}{(b - wL_c)^2 z} \times$$

$$\left[L_p \left(1 + rz - e^{rz} - \frac{wL_c}{b} \cdot \frac{e^{rz}}{1 - \frac{b''}{b'r}} \right) + L_h \left(e^{-rz} + rz - 1 - \frac{wL_c}{b} \cdot \frac{1 + rz}{1 - \frac{b''}{b'r}} \right) \right]$$

The L_p term is always < 0.

The L_h term is > 0 for w close to 0. But it may become < 0 for larger w, as rz gets small. At the limit, where wL_c approaches b(z), and rz gets very small, using the fact that for small rz, $b' = V_o r = (b-wL_c)/z$:

$$\frac{dTN}{dw} = \frac{b}{(V_o)^2 z} \left[L_p \left(-\frac{1}{2} + \frac{V_o}{b'' z^2} \right) + L_h \left(\frac{1}{2} + \frac{V_o}{b'' z^2} \right) \right]$$

The smaller the value of z at this maximum point, for a given V_o and b", the more likely dTN/dw < 0. In the limit of maximum z close to 0, dTN/dw must be < 0, as in the instantaneous production models in earlier sections.

Effect of Better Quality Land, Lower Labor (Column 4)

The effect of lower labor requirements is found by taking the derivative $-\frac{d}{dL_c}$, or, where L_h and L_p appear with different coefficients, taking the negative of their derivatives separately: $-\frac{d}{dL_h}$ and $-\frac{d}{dL_p}$.

11. Rent Share: $RS = \frac{R}{Y} = \frac{rV}{Y}$.

Rent share can be written as a product of two terms:

$$RS = \frac{rz}{e^{rz} - 1} \cdot \frac{b - wL_h - wL_p e^{rz}}{b}$$

Both of these terms increase as L_h or L_p fall, and therefore z falls. So rent share rises as labor requirements fall.

15. Improvement Value: $IM = W - V = \frac{b - wL_h - wL_p}{rz} - \frac{b - wL_h - wL_p e^{rz}}{e^{rz} - 1}$

$$-\frac{dIM}{dL_p} = w\left(\frac{1}{rz} - \frac{1}{1 - e^{-rz}}\right) - \frac{1}{r}\frac{dP}{dz}\frac{dz}{dL_p} \quad < \quad 0$$

$$-\frac{dIM}{dL_h} = w\left(\frac{1}{rz} - \frac{1}{e^{rz} - 1}\right) - \frac{1}{r}\frac{dP}{dz}\frac{dz}{dL_h} \quad ?$$

$$= \frac{w}{2}\left[1 - \frac{1}{rz\left(1 - \frac{b''}{b'r}\right)}\right] < 0 \quad \text{for small } rz$$

$$= \frac{w}{rz}\left[1 - \frac{1}{1 - \frac{b''}{b'r}}\right] > 0 \quad \text{for very large } rz$$

16. Ratio of Improvement to Land Value: $RT = IM/V = W/V - 1$

$$-dRT/dL_b = -d/dL_b\,(W/V).$$

$$W/V = \frac{b - wL_h - wL_p}{b - wL_h - wL_p e^{rz}} \cdot \frac{e^{rz} - 1}{rz}$$

Both terms on the right fall as z falls, so W/V falls -- naturally, since the smaller rz, the closer W to V. (Though W does not converge to V except for the everywhere concave production function, where b(0) = 0.)

Effect of Better Quality Land, Higher Productivity (Column 5).

Increased productivity of land is modelled by increasing a constant, k, multiplying production as a function of cycle length: kb(z). So the formula for land value, for optimal cycle length, and others are modified by adding a "k" next to b, b', and b". The effect of increased productivity is then found by taking the derivative $\frac{d}{dk}$.

11. Rent Share: RS = R/Y.

$\frac{dRS}{dk} > 0$ by the same argument showing $-\frac{dRS}{dL_c} > 0$.

16. Ratio of Improvement to Land Value: RT = IM/V

See 16. for Column 4, above.

Comparative Advantage on Better Quality Land

Say that Large and Small own land of the same quality. Large has a comparative advantage on better quality land if the rate of increase in land value at that point is higher for Large than for Small.

a. Land with a lower labor requirement, L_p or L_h:

On the same quality land a lower labor requirement raises land value:

$$-\frac{dV}{dL_p} = \frac{w}{1 - e^{-rz}} > 0$$

$$-\frac{dV}{dL_h} = \frac{w}{e^{rz} - 1} > 0$$

The effect of greater wealth on the rate of increase is:

$$\frac{d}{dw}\left(-\frac{dV}{dL_p}\right)\Bigg|_{V_o} = \frac{1}{1 - e^{-rz}}\left[1 - \frac{w}{e^{rz} - 1}\frac{drz}{dw}\Bigg|_{V_o}\right] > 0$$

$$\frac{d}{dw}\left(-\frac{dV}{dL_h}\right)\Bigg|_{V_o} = \frac{1}{e^{rz} - 1}\left[1 - \frac{w}{1 - e^{-rz}}\frac{drz}{dw}\Bigg|_{V_o}\right] > 0$$

These derivatives must be > 0 for very small w, and they must be > 0 for larger w, since for larger w $\frac{drz}{dw} < 0$. So assume by continuity they are > 0 everywhere.

b. More productive land -- higher k, where output/cycle = kb(z). The rate of change in land value with k is:

$$\frac{dV}{dk} = \frac{b}{e^{rz} - 1} > 0$$

The rate of change in this value with w, for land of value V_o, is:

$$\frac{d}{dw}\left(\frac{dV}{dk}\right)\Bigg|_{V_o} = \frac{1}{e^{rz} - 1}\left[b'\frac{dz}{dw} - \frac{b}{1 - e^{-rz}}\frac{drz}{dw}\right] > 0$$

This derivative is obviously > 0 where $\frac{drz}{dw} < 0$, which is true for larger w, if not everywhere. For w = 0, the derivative reduces to:

$$\frac{d}{dw}\left(\frac{dV}{dk}\right)\Bigg|_{V_o, w = 0} = -\frac{1}{e^{rz} - 1}\frac{bz}{1-e^{-rz}}\frac{dr}{dw} = \frac{L_h + L_p e^{rz}}{k(e^{rz} - 1)} > 0$$

So assume by continuity that the derivative is > 0 everywhere.

Table 5.4 1
BUILDING MODEL

	Economic Measure	Effect of Greatr Wealth on Same Qualty Land: To Observer: d/dz	Effect of Greatr Wealth on Same Qualty Land: To Owner: d/dw $\|V_o$	Effect of Better Quality Land: Lower Labr: $-d/dL_b, dL_m$	Effect of Better Quality Land: Higher Prod: d/dk
1	Cycle length: z	$1 > 0$	$\dfrac{\frac{L_b V_o}{1-e^{-rz}} + \frac{L_m}{r}(V_o + r\frac{dV}{dr})}{f' \frac{dV}{dr}} > 0$*	$-\frac{dz}{dL_b} = \frac{rw}{f'(1-e^{-rz})} \leq 0$; $-\frac{dz}{dL_m} = 0$	$\dfrac{r\int_0^z f(x)e^{-rx}dx}{kf'(1-e^{-rz})} - \frac{f(z)}{kf'} \leq 0$
2	Cycle x discnt: rz	$r > 0$	< 0 mostly*	$-r\frac{dz}{dL_b} < 0$; $-\frac{d}{dL_m} = 0$	$r\frac{dz}{dk} < 0$
3	Outpt/cyc: $OP = \int_0^z f(x)dx$	$f(z) \geq 0$	$f(z)\frac{dz}{dw} > 0$	$-f(z)\frac{dz}{dL_b} < 0$; $= 0$	$\frac{OP}{k} + kf(z)\frac{dz}{dk} > 0$ *
4	Gross income/acre: $Y = OP/z$	$\frac{f(z) - Y}{z} < 0$	$\frac{dY}{dz}\frac{dz}{dw} < 0$	$-\frac{dY}{dz}\frac{dz}{dL_b} > 0$; $= 0$	$\frac{Y}{k} + \frac{dY}{dz}\frac{dz}{dk} > 0$
5	Labr/acre: $L = \frac{L_b}{z} + L_m$	$-\frac{L_b}{z^2} < 0$	$-\frac{L_b}{z^2}\frac{dz}{dw} < 0$	$-\frac{1}{z}(1 - \frac{L_b}{z}\frac{dz}{dL_b}) < 0$** mostly; $-1 < 0$	$-\frac{L_b}{z^2}\frac{dz}{dk} > 0$
6	Labr cost/acre: wL = Depreciation	$-\frac{wL_b}{z^2} < 0$	$L_m + \frac{L_b}{z}(1 - \frac{w}{z}\frac{dz}{dw}) > 0$** mostly	$-w\frac{dL}{dL_b} < 0$**; $-w < 0$	$-\frac{wL_b}{z^2}\frac{dz}{dk} > 0$
7	Rent/acre: $R = rV$	0	$V_o\frac{dr}{dw} = V_o(\frac{L_m}{r} + \frac{L_b}{1-e^{-rz}})/\frac{dV}{dr} < 0$	$-r\frac{dV}{dL_b}$; $-r\frac{dV}{dL_m}$, > 0	$r\frac{dV}{dk} > 0$
8	Profit/acre: $P = Y - wL$	> 0 up to z^*_{ob}, then < 0 *	$-(L_m + \frac{L_b}{z}) + \frac{dP}{dz}\frac{dz}{dw} < 0$ because $\frac{dP}{dz} < 0$ *	$\frac{w}{z} - \frac{dP}{dz}\frac{dz}{dL_b}$; w, > 0	$\frac{Y}{k} + \frac{dP}{dz}\frac{dz}{dk} > 0$

Table 5.4.2

	Economic Measure	Effect of Greatr Wealth on Same Qualty Land		Effect of Better Quality Land	
		To Observer: d/dz	To Owner: d/dw $\|V_o$	Lower Labr: $-d/dL_b; dL_m$	Higher Prod: d/dk
9	Av.prod.labor: $AP = \int_0^z \frac{f(x)dx}{L_b + L_m z}$	$\frac{L_b f(z) + L_m[f(z)z - \int_0^z f(x)dx]}{(L_b + L_m z)^2} > 0$ *	$\frac{dAP}{dz}\frac{dz}{dw} > 0$	$-\frac{dAP}{dL_b} - \frac{dAP}{dz}\frac{dz}{dL_b} > 0$ *; $-\frac{dAP}{dL_m} > 0$	$\frac{AP}{k} + \frac{dAP}{dz}\frac{dz}{dk} > 0$ *
10	Labor share: $LS = \frac{wL}{Y} = \frac{w}{AP}$	$-\frac{w}{(AP)^2}\frac{dAP}{dz} < 0$	$\frac{1}{AP}[1 - \frac{w}{AP}\frac{dAP}{dw}] > 0$ *	$\frac{w}{(AP)^2}\frac{dAP}{dL_b} < 0$; < 0	$-\frac{w}{(AP)^2}\frac{dAP}{dk} < 0$
11	Rent share: $RS = \frac{R}{Y}$	$-\frac{R}{Y^2}\frac{dY}{dz} > 0$	< 0 *	$\frac{rw}{Y(1-e^{-rz})} \cdot [1 + \frac{rV(Y-f(z))}{Yf'z}] > 0$ *	$\frac{r}{Y}[\frac{dV}{dk} - \frac{V}{k} - \frac{V}{Y}\frac{dY}{dz}\frac{dz}{dk}] > 0$*
12	Profit share: $PS = 1 - LS$	$-\frac{dLS}{dz} > 0$	$-\frac{dLS}{dw} < 0$	$\frac{dLS}{dL_b} > 0$	$-\frac{dLS}{dk} > 0$
13	Land value: $V = \int_0^z \frac{f(x)e^{-rx}dx - wL_b}{1 - e^{-rz}} - \frac{wL_m}{r}$	0	0	$\frac{w}{1-e^{-rz}}$; $\frac{w}{r} > 0$	$\int_0^z \frac{f(x)e^{-rx}dx}{1-e^{-rz}} > 0$
14	Total value: $W = \frac{P}{r}$	$\frac{1}{r}\frac{dP}{dz}$ or $\frac{dW}{dz} < 0$ *	> 0 sm z, then ? *	$-\frac{1}{r}\frac{dP}{dL_b}$; $-\frac{1}{r}\frac{dP}{dL_m} > 0$	$\frac{1}{r}\frac{dP}{dk} > 0$
15	Improvement value: $IM = W - V$	$\frac{1}{r}\frac{dP}{dz}$ or $\frac{dW}{dz} < 0$	$\frac{dW}{dw}$	$-\frac{dIM}{dL_b} < 0$* sm z; $-\frac{dIM}{dL_m} = 0$*	$\frac{IM}{k} + \frac{dIM}{dz}\frac{dz}{dk}$? *
16	Ratio impr to land value: $RT = IM/V$	$\frac{1}{rV_o}\frac{dP}{dz}$ or $\frac{1}{V_o}\frac{dW}{dz} < 0$	$\frac{1}{V_o}\frac{dW}{dw}$	$-\frac{dRT}{dL_b} < 0$*; $-\frac{dRT}{dL_m} < 0$	$\frac{dRT}{dk} < 0$ *
17	Capital turnover: $TN = \frac{Y}{W} = \frac{r}{PS}$	$-\frac{r}{(PS)^2}\frac{dPS}{dz} < 0$ *	$\frac{1}{W}[\frac{dY}{dw} - \frac{Y}{W}\frac{dW}{dw}] < 0$ *	$\frac{r}{(PS)^2}\frac{dPS}{dL_b} < 0$; $\frac{r}{(PS)^2}\frac{dPS}{dL_m} < 0$	$-\frac{r}{(PS)^2}\frac{dPS}{dk} < 0$

Footnotes to Table 5.4.

* Discussed further in notes to Table 5.4.

** Sign depends upon elasticity of z wrt w or L_b, $\frac{w}{z}\frac{dz}{dw}$ and $\frac{L_b}{z}\frac{dz}{dL_b}$. These elasticities will be < 1 for small w or L_b and < 1 in general for most functions. The elasticity for large w or L_b, near maximum z, depends on the rate at which f(t) is falling, that is, on the magnitude of - f'. The greater the rate of decrease, that is, the larger - f', the smaller the elasticity.

5.7 Notes to Table 5.4

Effect of Greater Wealth on Same Quality Land, To Observer (Col. 2)

The observer notices only that cycle length increases with wealth. He imputes the same wage and discount rate to all landowners.

8. Profit per acre: $P = Y - wL$

For the observer, there is a value of z, z^*_{ob}, where profit is maximum. For very small rz, $z^*_{ob} = z_{ob}$; for larger rz, $z^*_{ob} < z_{ob}$. So to the observer, the profit per acre of landowners on the same quality land rises with the landowners' wealth until the landowners' cycle length reaches z^*_{ob}; then it falls again.

14. Total Value per acre: W

As discussed in notes to Table 3, there are two possible assumptions the observer may make in measuring total value.

a. The observer may assume landowners will continue the same cycle length in the future. In this case:

$$\frac{dW}{dz} = \frac{1}{r}\frac{dP}{dz}$$

Observed total value rises and falls with observed profit.

b. The observer may assume that the "correct" cycle length, z_{ob}, applies in the future. This assumption gives him the common sense result that an older collection of buildings is less valuable.

Formally, assume there are z cells of land, labelled s = 1...z. The buildings on each cell are s years old. The value of each cell must be:

$$W_s = \int_s^{z_{ob}} (f(x)-wL_m)e^{-r(x-s)}dx + V_o e^{-r(z_{ob}-s)} \qquad s \leq z_{ob}$$

$$W_s = V_o \qquad s \geq z_{ob}$$

The observer assumes that buildings under z_{ob} years old will be kept until they reach z_{ob}; buildings over z_{ob} will be immediately demolished.

So for $z \leq z_{ob}$, and $z \geq z_{ob}$, $W_<$ and $W_>$ are:

$$W_< = \frac{1}{z}\int_0^z W_s ds$$

$$= \frac{1}{rz}\Big[\int_0^z (f(x)-wL_m)(1-e^{-rx})dx + \int_z^{z_{ob}}(f(x)-wL_m-V_o r)(e^{-r(x-z)}-e^{-rx})dx$$

$$+ (1-e^{-rz})V_o \Big]$$

Straightforward differentiation or even inspection shows that $W_<$ falls as z increases.

$$W_> = \frac{1}{z}\int_0^{z_{ob}} W_s ds \qquad \text{for } s \leq z_{ob}$$

$$+ \frac{V_o}{z}(z - z_{ob}) \qquad \text{for } s > z_{ob}$$

$$= \frac{1}{rz}[\, rz_{ob}\ IM_{ob}\,] + V$$

-- where IM_{ob} is improvement value at z_{ob}.

$W_>$ obviously falls as z increases.

So total value perceived by an observer falls as wealth increases. Since land value is assumed constant, improvement value and ratio of improvement to land value also fall.

17. Capital Turnover: $TN = Y/W$ or r/PS

As in the tree model, the observer will get two different values for turnover, depending on what he assumes, so that $Y/W \neq r/PS$. If he measures turnover as r/PS, it will fall as wealth increases. If he measures W as above, it is not obvious that TN will fall with wealth, as both the numerator and the denominator of Y/W fall.

However, by substituting in the expressions for $W_<$ and $W_>$, it is easy to show that TN does in fact fall:

For $W_<$, $TN_< = Y/W_<$, and:

$$\frac{dTN_<}{dz} = rV_o \left[\frac{f(z)}{V_o} - TN_< \right]$$

$$= 0 \text{ at } z_{min}$$

$$= rV_o \left[\frac{f(z_{ob})}{V_o} - TN_{ob} \right] < 0 \text{ at } z = z_{ob}$$

For $W_>$,

$$TN_> = \frac{Y}{W_>} = \frac{\int_0^z f(x)dx}{z_{ob}IM_{ob} + zV_o}$$

It is apparent from inspection or easy differentiation that $TN_<$ falls as z increases.

Effect of Greater Wealth on Same Land, To Owner (Column 3).

Since it is assumed that Large has a higher wage, $\left.\frac{d}{dw}\right|_{V_o}$ ($\frac{d}{dw}$ for short) gives the direction of change with increased wealth.

1. Cycle Length and Discount Rate: z and r.

As described in 5.4, the requirement that Large and Small own the same quality land, which must have the same value to both, adds a constraint (4.9) to the equation for optimal cycle length (4.2).

These two equations combined yield formulas for change in discount rate, r, and in cycle length, z, with increased w:

$$(4.10) \qquad \left.\frac{dr}{dw}\right|_{V_o} = \frac{\frac{L_m}{r} + \frac{L_b}{(1 - e^{-rz})}}{\frac{dV}{dr}} < 0$$

The sign for $\left.\frac{dr}{dw}\right|_{V_o}$ is determined by the sign of $\frac{dV}{dr}$:

$$\frac{dV}{dr} = -\frac{Vze^{-rz}}{1 - e^{-rz}} - \int_0^z \frac{(f(x)-wL_m)xe^{-rx}dx}{1 - e^{-rz}} < 0$$

So discount rate falls as wage increases.

The formula for change in cycle length is:

$$(4.11) \qquad \left.\frac{dz}{dw}\right|_{V_o} = \frac{1}{f'\frac{dV}{dr}} \left[\frac{L_b V_o}{1 - e^{-rz}} + \frac{L_m}{r}\left(V_o + r\frac{dV}{dr}\right) \right]$$

$$= \frac{L_b V_o + L_m \left[\frac{f(z)(1-e^{-rz}(1+rz))}{r^2} - \int_0^z f(x)xe^{-rx}dx \right]}{f'\frac{dV}{dr}(1 - e^{-rz})}$$

The sign of this equation is not obvious on inspection, because the coefficient of L_m is ≤ 0. That is:

$$(7.1) \qquad \frac{f(z)[1 - e^{-rz}(1-rz)]}{r^2} - \int_0^z f(x)xe^{-rz}dx \leq 0$$

$$= 0 \text{ for } f(x) = f(z) = \text{const}$$

However, the sign can be shown ≥ 0 by considering the limiting points, $w = 0$, and w_{max}, where r gets very small.

For $w = 0$, z is a minimum, z_{min}. For a continually declining function, $z_{min} = 0$; for a function that is perfectly level and then declines, z_{min} is the point where the decline starts. In either case, $f(0) = f(z_{min}) = f(x)$, so that the coefficient of L_m, (7.1), = 0. So, provided $L_b > 0$, $\left.\frac{dz}{dw}\right|_{V_o, z_{min}} > 0$.

For w approaching a maximum, r must get very small, so that V_o remains constant. So rz gets very small. Then equations (4.2) and (4.9) can be solved, eliminating V_o, L_m, and r, to obtain:

$$(7.2) \qquad f(z)z - \int_0^z f(x)dx + wL_b = 0$$

From which:

$$(7.3) \qquad \left.\frac{dz}{dw}\right|_{V_o, w_{max}} = -\frac{L_b}{f'z} > 0$$

So if $\left.\frac{dz}{dw}\right|_{V_o} > 0$ at both ends of the range of w, by continuity it is > 0 in between as well. This implies in turn that $z = z_{max}$ at w_{max}.

It is also possible to solve (4.2) and (4.9) at w_{max}, z_{max} to obtain:

$$(7.4) \qquad \frac{L_b}{L_m} = \frac{\int_0^{z_{max}} f(x)dx - f(z_{max})z_{max}}{f(z_{max})}$$

From (7.4) it is apparent that the smaller the ratio L_b/L_m, the smaller z_{max}. For $L_b = 0$, z_{max} must $= z_{min}$. So if $L_b = 0$, $\left.\frac{dz}{dw}\right|_{V_o} = 0$, -- as is the case without the constraint V_o = const.

On the other hand, the larger the ratio L_b/L_m, the smaller $f(z_{max})$ and the larger z_{max}. When $L_m = 0$, z_{max} must be such that $f(z_{max}) = 0$. This is the maximum z_{max}. So the range of z is greatest when $L_m = 0$.

2. Cycle Length x Discount Rate: rz

rz obviously falls near z_{max} as w increases. For z reaches a definite maximum, while r becomes indefinitely small as w increases --

as is apparent from (4.3): $f(z) - wL_m = V_o$.

What happens at z_{min}, where $w = 0$?

$$\left.\frac{drz}{dw}\right|_{V_o, z_{min}} = \frac{r\left[\frac{f'z}{r}\left[L_b + \frac{L_m(1-e^{-rz})}{r}\right] + L_b V_o\right]}{f' \frac{dV}{dr}(1 - e^{-rz})}$$

The f'z term in the numerator is < 0, if $z_{min} > 0$; or $= 0$ if $z_{min} = 0$. The L_bV_o term is > 0. So if $z_{min} = 0$, rz starts from 0, increases, and then approaches 0 again as r becomes very small. If $z_{min} > 0$, rz may increase or decrease at first, depending on the relative size of the f'z and L_bV_o terms. But it eventually declines towards 0.

<u>8. Profit per acre</u>: $P = Y - wL$

$$\frac{dP}{dz} = \frac{dY}{dz} - w\frac{dL}{dz}$$

$$= \frac{1}{z}\left[f(z)\frac{rz - (1-e^{-rz})}{rz} - \frac{1}{z}\int_0^z f(x)(1-e^{-rx})dx \right] < 0$$

Notice that the second equation, obtained by substituting from (4.2), the equation for optimal z, is not correct for the observer since the cycle lengths he perceives are not optimal for him. To the owner, profit should be falling at optimal z, since the profit-maximizing cycle length z* is too short.

<u>9. Average Product of Labor:</u> $AP = Y/L$

$$AP = \frac{1}{L_b + L_m z}\int_0^z f(x)dx$$

$$\left.\frac{dAP}{dw}\right|_{V_o} = \frac{dAP}{dz}\left.\frac{dz}{dw}\right|_{V_o} = \frac{\left[L_b f(z) + L_m \left[f(z)z - \int_0^z f(x)dx\right] \right]}{(L_b + L_m z)^2}\left.\frac{dz}{dw}\right|_{V_o}$$

This expression is obviously > 0 at z_{min}, because at z_{min} the coefficient of $L_m = 0$. For larger z, the coefficient is < 0.

At z_{max}, the coefficient of L_m equals $- f(z_{max})L_b/L_m$, from (7.4). Substituting this into the expression for dAP/dz shows that:

$$\left.\frac{dAP}{dz}\right|_{V_o, z_{max}} = 0$$

So assume by continuity that dAP/dz is > 0 everywhere up to z_{max}. In fact, AP ranges from $f(z_{min})z_{min}/(L_b + L_m z_{min})$, which = 0 for z_{min} = 0, to a maximum of $\dfrac{\int_0^{z_{max}} f(x)dx - f(z_{max})z_{max}}{L_b}$

<u>10. Labor Share of Output:</u> LS

$$LS = \frac{wL}{Y} = \frac{w}{AP}$$

$$\left.\frac{dLS}{dw}\right|_{V_o} = \frac{1}{(AP)^2}\left[AP - w \frac{dAP}{dz} \left.\frac{dz}{dw}\right|_{V_o} \right]$$

This expression is obviously > 0 at z_{min}, where w = 0; and at z_{max}, where dAP/dz = 0. Assume by continuity it is > 0 everywhere.

<u>11. Rent Share of Output:</u> RS

$$RS = \frac{rV_o}{Y} = \frac{rzV_o}{\int_0^z f(x)d\,x} = \frac{rzV_o}{OP}$$

Rent share obviously falls as w increases where rz is falling, since OP, output per cycle, increases as z increases.

But what if rz is increasing for small w and small z?

At z_{min}, $Y = f(z_{min})$, so dY/dz = 0. Since dr/dw < 0 everywhere, RS must fall as w increases, even at z_{min}. So assume by continuity that RS falls everywhere as w increases.

<u>14. & 15. Total Value, and Improvement Value:</u> $W = P/r$; $IM = W - V_o$

$$W = \frac{1}{rz}\int_0^z (f(x)-wL_m)(1-e^{-rx})dx + V_o\frac{(1-e^{-rz})}{rz}$$

$$IM = \frac{1}{rz}\int_0^z (f(x)-wL_m)(1-e^{-rx})dx - V_o\frac{rz-(1-e^{-rz})}{rz}$$

Since V_o is constant,

$$\left.\frac{dW}{dw}\right|_{V_o} = \left.\frac{dIM}{dw}\right|_{V_o}$$

The expression for IM can be rewritten:

$$IM = \frac{1}{rz}\int_0^z f(x)(1-e^{-rx})dx - \frac{f(z)(rz-(1-e^{-rz}))}{r^2z}$$

So IM obviously equals 0 at z_{min}.

At z_{max}, where r approaches 0, IM becomes:

$$IM = \frac{1}{z}\int_0^z f(x)xdx - \frac{f(z)z}{2} > 0$$

IM can = 0 everywhere only for an instantaneous production function, such that $z_{min} = z_{max} = 0$.

$$\frac{dIM}{dw} = \left[-\frac{IM}{z} - \frac{f'(rz-(1-e^{-rz}))}{r^2z}\right]\frac{dz}{dw}$$

$$-\frac{1}{r^2z}\left[\int_0^z f(x)(1-e^{-rx}-rxe^{-rx})dx - \frac{f(z)(rz-2+(2+rz)e^{-rz})}{r}\right]\frac{dr}{dw}$$

The dr/dw term = 0 at z_{min}; otherwise it is > 0.

The dz/dw term is > 0 at z_{min}, where IM = 0, leaving only the positive – f' part. (If $z_{min} = 0$, $dIM/dw = L_b/2$.)

For larger z, the dz/dw term is ambiguous.

So W and IM may rise everywhere as w increases, or they may rise and then fall again. W starts out = V_o and IM starts out = 0, but if they fall again, they do not get back to V_o and 0.

17. Capital Turnover: $TN = Y/W = r/PS$

Since Y decreases everywhere for $z > z_{min}$, TN is clearly decreasing everywhere that W is increasing. The only possible problem arises for functions for which W may fall again towards z_{max}.

Notice that if $L_b = 0$, so that $dz/dw = 0$, IM and W can only increase as w increases, so that TN can only fall. So for simplicity, consider only the opposite case, $L_m = 0$:

$$\frac{dTN}{dw} = \frac{1}{W}\left[\frac{dY}{dw} - \frac{Y}{W}\frac{dW}{dw}\right]$$

$$= \frac{L_b}{W\frac{dV}{dr}(1-e^{-rz})rz} \cdot \left[-\frac{(V_o)^2 r(TN-r)}{f'} + \frac{TN}{r}\left[\int_0^z f(x)(1-e^{-rx}-rxe^{-rx})dx + V_o(1-e^{-rz}-rze^{-rz})\right]\right]$$

The first term in the double bracket is > 0, since TN > r, (if wL_b > 0). (The second term is always > 0, except for rz = 0, where it = 0.)

So $dTN/dw < 0$.

Effect of Better Quality Land, Lower Labor Requirements L_b & L_m (Col. 4)

Better quality land with lower labor requirements is modelled by taking the derivatives: $-\frac{d}{dL_b}$ and $-\frac{d}{dL_m}$. Note that $-\frac{dz}{dL_m} = 0$.

9. Average Product of Labor: $AP = Y/L$

$$-\frac{dAP}{dL_b} = \frac{AP}{L_b + L_m z} - \frac{dAP}{dz}\frac{dz}{dL_b}$$

$$\geq \frac{AP}{L_b + L_m z} - \frac{L_b f(z)}{(L_b + L_m z)^2}\frac{dz}{dL_b}$$

(The inequality comes from the fact that the L_m term of $dAP/dz < 0$.)

The expression to the right of the inequality ≥ 0 near z_{min}, where $w = 0$, because $-dz/dL_b = 0$ at that point, and $AP = 0$ too, if $z_{min} = 0$. It is > 0 near maximum z_{max}, where V and $L_m = 0$, because $f(z) = 0$ at that point. Assume by continuity that it is > 0 everywhere in between.

11. Rent Share: $RS = R/Y = rV/Y$

$$-\frac{dRS}{dL_b} = \frac{rw}{Y(1 - e^{-rz})}\left[1 + \frac{rV(Y - f(z))}{Yf'z}\right]$$

This expression $= 0$ at z_{min}, where $w = 0$, because at z_{min} $Y = f(z)$. It is also > 0 near z_{max}, because at z_{max}, $V = 0$. So assume it is > 0 everywhere in between.

15. Improvement Value/acre: $IM = W - V$

$$IM = \frac{1}{rz}\int_0^z f(x)(1-e^{-rx})dx - \frac{f(z)(rz - (1-e^{-rz}))}{r^2 z}$$

$$-\frac{dIM}{dL_b} = -\left[-IM - \frac{f'[rz - (1-e^{-rz})]}{r^2 z}\right]\frac{dz}{dL_b}$$

This expression is clearly < 0 for very small z, since $IM = 0$ at z_{min}, leaving only the positive $-f'$ term times $-dz/dL_b$.

However, the expression is ambiguous for larger z, where $IM > 0$.

$- dM/dL_m = 0$, because $- dz/dL_m = 0$.

16. Ratio of Improvement to Land Value: RT = M/V

The ratio falls with a fall in L_b or L_m. This is apparent from considering the limits. At minimum z, IM and therefore RT = 0. Near z_{max}, where V goes to 0, the ratio becomes indefinitely large. Therefore, a decrease in z, due to a decrease in L_b, or an increase in V, due to a decrease in L_b or L_m, will reduce RT.

The Effect of Better Quality Land, Higher Productivity: Higher k (Col. 5)

Higher productivity land is modelled by inserting a multiplicative factor, k, next to f(x), f(z) and f' in all expressions, and taking the derivative d/dk to give the effect of higher productivity.

3. Output/cycle: OP

$$OP = k\int_0^z f(x)dx$$

$$\frac{dOP}{dk} = \frac{OP}{k} + kf(z)\frac{dz}{dk}$$

This expression is > 0 at z_{min}, because at z_{min}, $dz/dk = 0$. It is also > 0 at maximum z_{max}, where L_m and V = 0, because here f(z) = 0. So assume dOP/dk > 0 everywhere.

9. Average Product of Labor: $AP = Y/L$

$$AP = \frac{k\int_0^z f(x)dx}{L_b + L_m z}$$

$$\frac{dAP}{dk} = \frac{AP}{k} + \frac{dAP}{dz}\frac{dz}{dk}$$

$$> \frac{AP}{k} + \frac{kL_b f(z)}{(L_b + L_m z)^2}\frac{dz}{dk}$$

The expression to the right of the inequality ≥ 0 at z_{min}, where $dz/dk = 0$, and $AP = 0$ if $z_{min} = 0$. It is > 0 near maximum z_{max}, where L_b and $V = 0$, because here $f(z) = 0$. So assume it is > 0 everywhere in between.

11. Rent Share: $RS = R/Y = rV/Y$

$$\frac{dRS}{dk} = \frac{r}{Y}\left[\frac{dV}{dk} - \frac{V}{k} - \frac{V}{Y}\frac{dY}{dz}\frac{dz}{dk}\right]$$

This expression $= 0$ at z_{min}, where $RS = 1$. It is > 0 at z_{max}, where $V = 0$. So assume it is > 0 everywhere in between.

15. Improvement Value: $IM = W - V$

$$\frac{dIM}{dk} = \frac{IM}{k} + \frac{dIM}{dz}\frac{dz}{dk}$$

The effect of increased k on IM is ambiguous. An increase in k raises improvement value for a given cycle length, but shortens cycle length, which reduces improvement value.

16. Ratio of Improvement to Land Value: $RT = IM/V$

The ratio falls as k increases. This is again apparent from considering the limits. For z_{min}, IM = 0. For z_{max}, V = 0. Therefore, assume by continuity that since z falls as k increases, RT falls too.

Comparative Advantage on Better Quality Land

Say that Large and Small own land of the same quality. Large has a comparative advantage on better quality land if the rate of increase in land value at that point is higher for Large than for Small.

a. Land with lower labor requirement, L_b or L_m

$$-\frac{dV}{dL_b} = \frac{w}{1 - e^{-rz}}$$

$$-\frac{dV}{dL_m} = \frac{w}{r}$$

$$\frac{d}{dw}\left(-\frac{dV}{dL_b}\right)\bigg|_{V_o} = \frac{1}{1 - e^{-rz}}\left[1 - \frac{w}{e^{rz} - 1}\frac{drz}{dw}\right]$$

This expression must be > 0 near z_{min}, where w = 0. It also must be > 0 near z_{max}, since here $drz/dw < 0$. So assume by continuity it is > 0 everywhere.

$$\frac{d}{dw}\left(-\frac{dV}{dL_m}\right)\bigg|_{V_o} = \frac{1}{r} - \frac{w}{r^2}\frac{dr}{dw} > 0$$

b. More productive land -- higher k, where output/cycle = $k\int_0^z f(x)dx$

$$\frac{dV}{dk} = \frac{\int_0^z f(x)e^{-rx}dx}{1 - e^{-rz}} = \frac{1}{k}\left[V + \frac{wL_b}{1-e^{-rz}} + \frac{wL_m}{r}\right]$$

$$\frac{d}{dw}\left(\frac{dV}{dk}\right)\bigg|_{V_o} = \frac{L_b}{k(1-e^{-rz})}\left[1 - \frac{w}{1-e^{-rz}}\frac{drz}{dw}\right] + \frac{L_m}{r}\left[1 - \frac{w}{r}\frac{dr}{dw}\right]$$

The first term in square brackets is identical to the term in square brackets in the expression for $d/dw\ (- dV/dL_b)|V_o$, and is > 0 by the same line of reasoning. The second term in brackets is > 0 because $dr/dw < 0$.

So Large has a comparative advantage on better quality land, whether land with lower labor requirements, or more productive land.