CHAPTER 1

EFFECT OF WEALTH OR FIRM SIZE, WITH AND WITHOUT TRANSACTIONS COSTS

1.1 Summary[A]

How do richer people differ from poorer, or bigger firms from smaller, with and without transactions costs? Here's a simple model:

In a remote corner of the Land of Oz lies the country of the Clones. The Clones are farmers. As their name implies, the Clones resemble each other down to the last quirk in their utility functions--which depend on food and leisure only. Likewise their farmland--where production depends only on land and labor--is all the same quality. However, Clonelanders may own all different sizes of farms, or no land at all.

The Clones may work on their own land, if any, and do so unless very rich. They may also hire or be hired by other Clones at the market wage. But a Clone who hires additional labor must spend some of his own labor time supervising at a given rate. So transactions costs take the form of a supervision requirement.

Given these assumptions, the Clones logically divide up into four categories according to the size range of their land:

The "peasants", owning little or no land, farm their own land (if any) and hire themselves out at the going wage to richer landowners.

The "self-sufficient farmers", owning somewhat more land, farm their own land but neither hire nor are hired by others--as the supervision requirement makes neither profitable.

The "small landlords", owning even more land, hire and supervise peasants at the going wage and work alongside them.

The "large landlords", owning the biggest farms, do not personally touch the soil, but merely supervise hired peasants.

Cloneland is a perfect place to test the effects of differences in distribution of wealth or in the level of transactions costs (the rate of supervision required). Chapter 1 works out the "micro" effects of such differences on the four groups, peasants, self-sufficient farmers, small landlords and large landlords. Chapter 7 will add up these micro effects to find the general equilibrium effects.

The self-sufficient farmers.

The self-sufficient farmers provide the paradigm of differences in wealth (land size), given transactions costs. It's also possible to look at the farmers as a collection of different size firms, constrained by transactions costs to hire their owners' labor. So differences between richer and poorer farmers are equally differences between larger and smaller firms.

Here are the most salient differences: The richer the farmer, or larger the firm, the higher the farmer's implicit wage. Assuming no backwards bending labor supply curves, that means the richer the farmer, the lower his ratio of labor to land; hence, the lower his output per acre and the marginal product of his land, but the higher his average product of labor. Column A of Table 1.1 summarizes these and other results for the self-sufficient farmer.

The prediction that average product of labor rises with wealth or firm size is particularly useful and testable. For this prediction holds up under a wide variety of complications introduced in later chapters. Moreover, average product of labor is both easy to measure, and of all measures, least subject to distortion by conventional (or unconventional) accounting methods. And in fact reams of data confirm a rise in average product of labor with wealth or firm size. (Eg. look at

Table 1.1

Effect of Increased Land Size, or Increased Supervision Rate*

A: Landowner works directly on his land; does not hire or work for hire, (Sec. 1.5, Table 1.4).

B: Landowner works on own land, and works for hire at given wage, (Sec. 1.6, Table 1.5).

C: Landowner works directly on land and supervises hired labor, (Sec. 1.7, Table 1.6).

D: Landowner only supervises hired labor, (Sec. 1.8, Table 1.7).

	More Land				More Supervsion	
	A	B	C	D	C	D
1. Labor:						
Landowner's total	+	-	-	+	+	?
Landowner's applied	+	+	-		+ mstly	
Hired (or hired out,B)		-	+	+	- mstly	-
Applied (lndr's & hird	+	+	+	+	-	-
Total (applied & supr)	+	+	+	+	- mstly	?
2. Ratio, labor to land:						
Applied	-	0	0	-	-	-
Total	-	0	+	-	- mstly	?
3. Wage, MP labor:						
Employee's wage			0	0	0	0
Landowner's wage	+	0	0	+	+	+(?)
MP applied labor	+	0	0	+	+	+
Weighted average wage			-	+	+	?
4. Labor cost: = wage x total labor (in firm)	+	+	+	+	+then-?	?
5. Labor cost/acre:	+ then-	0	0	+?then-	?	?
6. Output:	+	+	+	+	-	-

Table 1.1, continued

	More Land				More Supervsion	
	A	B	C	D	C	D
7. MP land: = Profit/acre (const. returns only)	-	0	0	-	-	-
8. AP labor: Output/applied labor	+	0	0	+	+	+
Output/total labor	+	0	-	+	+	?
9. AP land: = output/acre	-	0	0	-	-	-
10. Labor share: = labor cost/output	+	0	0	+	+	+
11. Profit: = output - labor cost	+ then-	+	+	+?then-	-	-
12. Landowner's income: = profit + time value	+	+	+	+	+ or -	?
13. Landowner's consptn: = profit + wages	+	+	+	+	?	-
14. Landowner's utility: (fctn of consumption and leisure)	+	+	+	+	-	-

* Assuming constant returns to scale in the underlying production function, to avoid minor complications.

any Fortune 500).

Ample data also indicates that wages for comparable work rise with firm size [eg. Lester, 1967], while intensity of resource use falls [eg. Martin, 1967]. Both facts are usually attributed to monopoly; the self-sufficient farmers suggest a more general and profound explanation.

Moreover, due to transactions costs, the self-sufficient farmers dodge the old returns-to-scale dilemma that vexes the neoclassical theory of the firm. This is the dilemma: On the one hand, if production does not show constant returns to scale, the payments to the factors of production do not equal the product--with increasing returns the payments exceed the product, while with decreasing returns they fall below the product. If matter cannot be created or destroyed, where does the deficit come from, or the surplus go? But on the other hand, under the unlikely assumption that all production shows constant returns to scale, there are two unpleasant possibilities: 1. Production technology is linear homogeneous, which leaves firm size totally indeterminate. 2. More plausibly, technology shows increasing and then decreasing returns to scale, with a point of constant returns in between. Then all firms in an industry must be the same size, the size at constant returns.

With transactions costs, this nasty dilemma vanishes. The self-sufficient farmers' firms may show increasing or decreasing returns in their underlying technology as a function of land and labor. But the deficit or surplus over rent and wages automatically goes to the farm owner as part of his firm's profit. Alternatively, the farmers can be described as experiencing net diseconomies of scale in land size--since with transactions costs, a farmer's wage and labor supply depend on land size. The surplus just goes to the owner, and all is well in Cloneland.

The Peasants

The peasants farm their own land (if any) and also work for hire at a given wage. Considered in isolation, the peasants predict the effects of wealth differences absent transactions costs. The silliness of these predictions further confirms the plausibility of the self-sufficient farmers' model.

Most notably, because all peasants work at the given market wage, differences in wealth can only have "income" effects. So the richer the peasant, the less he works! Ranging from landless peasants to richer and richer ones, labor supply falls slowly and then more and more rapidly. Not even Veblen's The Theory of the Leisure Class envisions such a plunge in effort with increasing wealth.

The peasants of course can say nothing about the effects of differences in firm size. For absent transactions costs, and necessarily assuming constant returns to scale, there can be no effects. Firms are either indeterminate in size or all the same size and identical. In fact there can be no ownership of firms in the operative sense that a person customarily works with and derives income from "his" specific property. A uniform smear of labor simply spreads across the peasants' land like butter on bread, at a ratio of labor to land determined solely by the market wage. The market wage likewise fixes the average and marginal products of labor and land on all peasant land.

Table 1.1 Col. B summarizes the results for the peasants.

The Landlords

The small and large landlords show not only the effects of wealth differences but also, explicitly, the effects of varying levels of transactions costs.

The large landlord only supervises hired labor. His wage exceeds the marginal product of labor on his land, which in turn exceeds the market wage paid to employees. Differences in wealth affect large landlords just as they do self-sufficient farmers. In fact, if the large landlords' farms are assumed to produce food net of payments for hired labor or, as shown in Chp. 2, net of payments to rented land--the large landlords become mathematically identical to the self-sufficient farmers. So the self-sufficient farmer model applies quite generally to individuals or firms that hire outside factors of production in a world with transactions costs. It's sufficient that there be a person--an owner or top manager--whose labor input is crucial and, there being only 24 hours in a day, in increasingly short supply as firm size increases.

The small landlords, who both work and supervise hired labor, show some interesting peculiarities, discussed in the text. Table 1.1, cols. C and D under More Land summarize the full results of wealth differences for both large and small landlords.

Interestingly enough, an increase in transactions costs (an increase in the required rate of supervision) has many of the same effects as an increase in wealth. For, as would be expected, an increase in transactions costs lowers the amount of labor applied per acre of land. So output per acre falls as transactions costs rise, while output per hour of applied labor rises. Columns C and D of Table 1.1 under "More Supervision" summarize these results.

1.2 Plan of Chapter 1[B]

Chapter 1 contains five small "models" of behavior.

Section 1.3 lists and explains the quite conventional assumptions of the models.

The first model, Section 1.4, depicts the "consumer-laborer", whose utility depends only on food and leisure. It derives his labor supply as a function of wage and income. Table 1.3 summarizes the behavior of the consumer-laborer.

The consumer-laborer can be combined with a land-owning firm, whose output "food" depends only on land size and quantity of labor applied. Depending on land size, there are four possible models, as noted above: "the self-sufficient farmer"--Sec. 1.5, "the peasant"--Sec. 1.6, "the small landlord"--Sec. 1.7, and "the large landlord"--Sec. 1.8. Tables 1.4 through 1.7 summarize the behavior of owners and their firms as a function of (exogenous) land size, and, where relevant, exogenous "market" wage for hired labor, and exogenous required rate of supervision of hired labor.

1.3 Assumptions of the Models[c]

(1) The Economy

The economy consists of an area of land populated by a large number of landowners. These landowners differ only in owning more or less land, or no land at all. Output depends on land and labor only; all land and labor is of the same quality. Landowners may hire labor from other landowners, at an exogenous "market" wage. But if they do hire labor, they must supervise at a given exogenous rate. The landowners msximize utility.

The landowners are treated as if each one owns a "firm". This firm owns the land, and hires the owner's and possibly others' labor. The firm maximizes profit, which it returns to the owner.

Consequently, differences between different size landowners are equally differences between different size firms.

(2) Time

There is only one time period. This is equivalent to assuming a steady state economy, with each time period identical to the one before. Models in later chapters generalize to many time periods.

(3) Utility Function and Labor Supply Curve

a. Everyone has the same utility function.

b. Utility depends on food and leisure only: $U = u(Q,Z)$, where Q is food and Z is leisure. This rules out, among other things, interdependent utility functions. Both food and leisure have positive utility: $u_1, u_2 > 0$. Consumption of leisure cannot exceed a maximum, D: the total number of hours in a day (or other time period).

c. Labor supply, L, is the difference between total time available

Table 1.2

Notation for Chapter 1

Q	food (and everything else but leisure) consumed
Z	leisure
D	total time available for leisure or labor
L	labor: $L \equiv D - Z$
H	hired labor
S	labor a landowner does himself on his own land
A	labor applied directly to land, (vs supervisory labor)
$U = u(Q,Z)$	utility as a function of food and leisure $u_1, u_2 > 0$.
1	price of food: unity
w	wage
v	market wage (exogenous)
P	profit
y	total income: $y = wD + P$
$L = a(y,w)$	labor supply as a function of total income and wage (see 1.4) $a_1 < 0$; $a_2 > 0$; $a_{11}, a_{22} < 0$; $a_{12} > 0$.
T	land (exogenous)
$F = f(T,L)$	output as a function of land and labor. Output is food, F, with unity price. $f_1, f_2, f_{12} > 0$; $f_{11}, f_{22} < 0$. $f_{11}f_{22} - (f_{12})^2 = 0$ linear homogeneous; < 0 economies of scale; > 0 diseconomies of scale
r	rent of land
k	required rate of supervision of hired labor: $0 \leq k \leq 1$ (exogenous)

for leisure, D, and consumption of leisure, Z: $L = D - Z$.

d. Given a total income, and a price for food and leisure, an individual maximizes utility subject to a budget constraint,--as in the Consumer-Laborer model, 1.4. Consequently, the demand for food and leisure, and the supply of labor, can all be written as functions of total income and prices of food and leisure.

e. The price of food remains constant, and equals unity, for convenience. The price of leisure necessarily equals the price of labor--the wage.

f. Food and leisure are "normal" goods--goods whose demand curve shifts out as income rises.

g. Labor supply is in fact a concave function of income and wage, rising asymptotically towards the absolute time limit, D, as wage increases, holding income constant, or income decreases, holding wage constant, as explained at length in Sec. 1.4.

h. The labor supply curve does not "bend backwards". That is, even if income increases as wage increases, the "income effect" of the higher wage does not outweigh the "price effect". In fact, assume in general that whenever income and wage rise simultaneously, labor supply rises--necessarily at a decreasing rate, given the limited number of hours in a day. This assumption does no harm, as the predictions of the models would be even stronger if labor supply curves could bend backwards. See Sec. 1.4 for discussion.

(4) Assumptions About Transactions and Transactions Costs

Only one transaction can occur in this simple economy: landowners (or their firms) may buy or sell labor at a given market wage rate, v.

If a landowner buys labor, he must supervise that labor at a fixed rate, k: $0 \leq k \leq 1$. So if he hires H hours of labor, he must spend kH hours of his own time supervising. Obviously, the greater the required rate of supervision, the less labor the landowner hires. When $k = 1$, he hires no labor, even at zero wage.

Consequently, transactions costs take an explicit and an implicit form:

The explicit form is the actual cost of the supervising landowner's labor: wkH, where w is the supervising landlord's wage.

The implicit form is the loss incurred when landowners fail to hire the extra labor they would have hired absent supervision requirements.

(5) Production and the Production Function

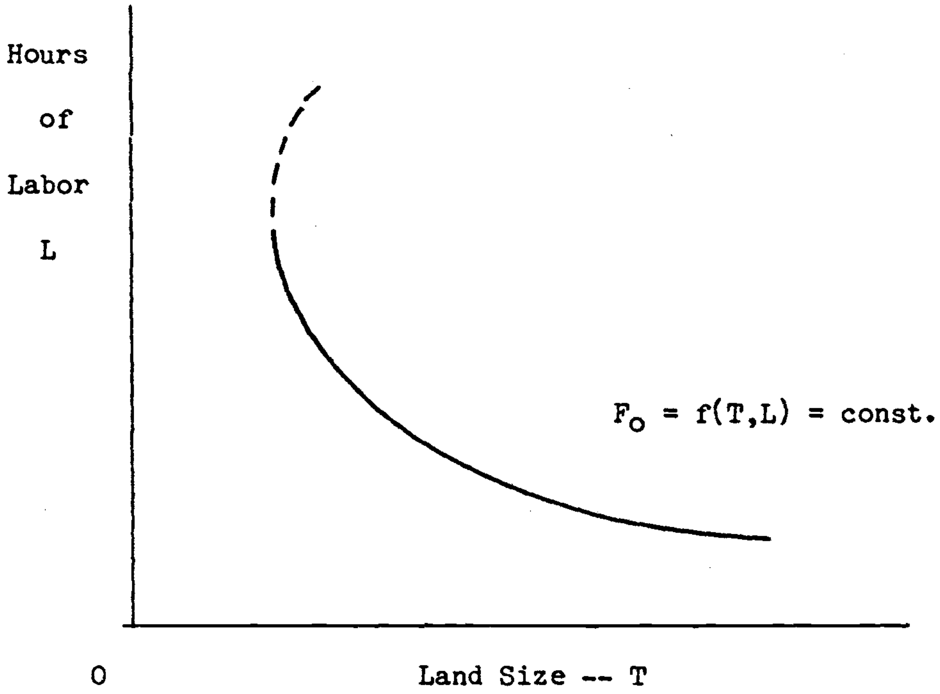
a. Everyone has the same native ability. So an hour of one person's labor is just as good as an hour of another's.

b. All land in the economy is the same quality.

c. There is only one output: F, "food", and one production function.

d. Production depends on land and labor only: $F = f(T,L)$, where T is land and L is labor. This production function behaves reasonably in the regions production occurs, that is: $f_1, f_2, f_{12} \geq 0$; and $f_{11}, f_{22} \leq 0$. Production is "instantaneous".

e. The production function isoquants are shaped as in Figure 1.1: The marginal product of labor, f_2, goes to zero when a large enough but finite quantity of labor crowds onto a given piece of land. As an immediate consequence, the payment to labor, f_2L, and the labor share of output, f_2L/f, must also go to zero. This common-sense assumption rules out CES production functions, including that intellectual Syren, the Cobb-Douglas function. Also assume, what usually amounts to the same thing, that the labor share of output always falls as the ratio of



'ig. 1.1: Isoquant: Output as a function of labor and land.

labor to land rises. Plausibly, the marginal product of land, f_1, goes to zero when the ratio of labor to land is small enough, but not zero. But assume only that f_1T falls as the ratio of labor to land falls.

These assumptions about the production function are unnecessary to any major results, but very helpful in working out details.

f. There may be economies or diseconomies of scale in the production function. Ie., $f_{11}f_{22} - (f_{12})^2 < 0$, and $f - f_1T - f_2L < 0$ for economies; and $f_{11}f_{22} - (f_{12})^2 > 0$, and $f - f_1T - f_2L > 0$ for diseconomies. Plausibly, most production functions show economies of scale at small scales, and diseconomies of scale at large enough scales. However, due to transactions costs, the landowner's firm must always experience net diseconomies of scale, at least beyond a certain minimum land size. That is, as land size increases, the effect of transactions costs must eventually outweigh economies of scale. Otherwise, the largest landowner hires all the labor of all the other landowners, --a ridiculous possibility.

1.4 The Consumer-Laborer[C]

This section develops the personal labor supply equation as a function of income and (exogenous) wage, and indirectly as a function of (exogenous) lump sum profit, under the assumption that the labor supply curve does not "bend backwards".

The Maximization Problem:

Suppose a person's utility depends only on food, Q, and leisure, Z, as described in 1.3.3. His labor supply L equals D - Z, where D is total time available for work or leisure. The price of food equals unity. The price of leisure and labor equals a given wage w. The person also receives a given profit, P, from a firm he owns. (P may be positive or negative, and may include lump-sum taxes.) Consequently, the person's consumption of food must equal wages for labor plus profit:

$$(4.1) \qquad Q = wL + P = w(D - Z) + P$$

And his total income must equal the value of food plus the value of leisure:

$$(4.2) \qquad y = Q + wZ = wD + P$$

This is his budget constraint. The person may be considered to sell all his time, D, to his firm at rate w, and then buy back wZ worth of leisure.

The person maximizes utility, subject to (4.2) and the requirement that $D - Z \geq 0$, that is, labor supply is non-negative.

The resulting Kuhn-Tucker Conditions are:

$$\frac{u_2}{u_1} - w \geq 0 \qquad (\frac{u_2}{u_1} - w)(D - Z) = 0$$

Labor Supply as a Function of Income, Wage and Profit:

If the equality holds, $u_2/u_1 - w = 0$ can be rewritten to give demand for food and leisure as a function of income, y, and wage, w. (The price of food is omitted as it is assumed constant and equal to one.) Since demand for leisure equals D minus the supply of labor, labor supply can be written:

(4.3) $L = a(y,w)$

From assumption 3f. that leisure is a normal good, it immediately follows that:

(4.4) $a_1 < 0$ "income effect": labor supply falls if income rises, holding wage constant.

(4.5) $a_2 > 0$ "wage effect": labor supply rises if wage rises, holding income constant.

From assumption 3g. that labor supply is a concave function of income and wage it follows that:

(4.6) $a_{11} < 0$

(4.7) $a_{22} < 0$

And from assumption 3g. that labor supply tends asymptotically towards the limit, D, as income falls, holding wage constant; or as wage rises, holding income constant, it follows that:

(4.8) $a_{12} > 0$

Figures 1.2 and 1.3 show labor supply as a function of income and wage.

Wage is exogenous, but income is not. But, given the budget constraint, (4.2), labor supply can be found as a function of wage and exogenous profit, instead of wage and income. So if wage increases holding profit constant, income increases too. The effect on labor supply must be:

(4.9) $$\left.\frac{dL}{dw}\right|_P = a_1 D + a_2 > 0$$

Labor supply must increase, even though $a_1 < 0$, by assumption 3h. that the labor supply curve does not bend backwards. Therefore, whenever the expression $a_1 D + a_2$ appears, we can assume it is > 0.

Figure 1.4 shows labor supply as a function of wage, holding profit constant.

Partial Derivatives with Respect to Wage and Profit

Table 1.3 below shows effects on labor supply, food demand, income, and utility of an increase in the exogenous variables, wage and profit.

Notice that, from the budget constraint, partials with respect to profit, holding wage constant, necessarily equal partials with respect to income, holding wage constant.

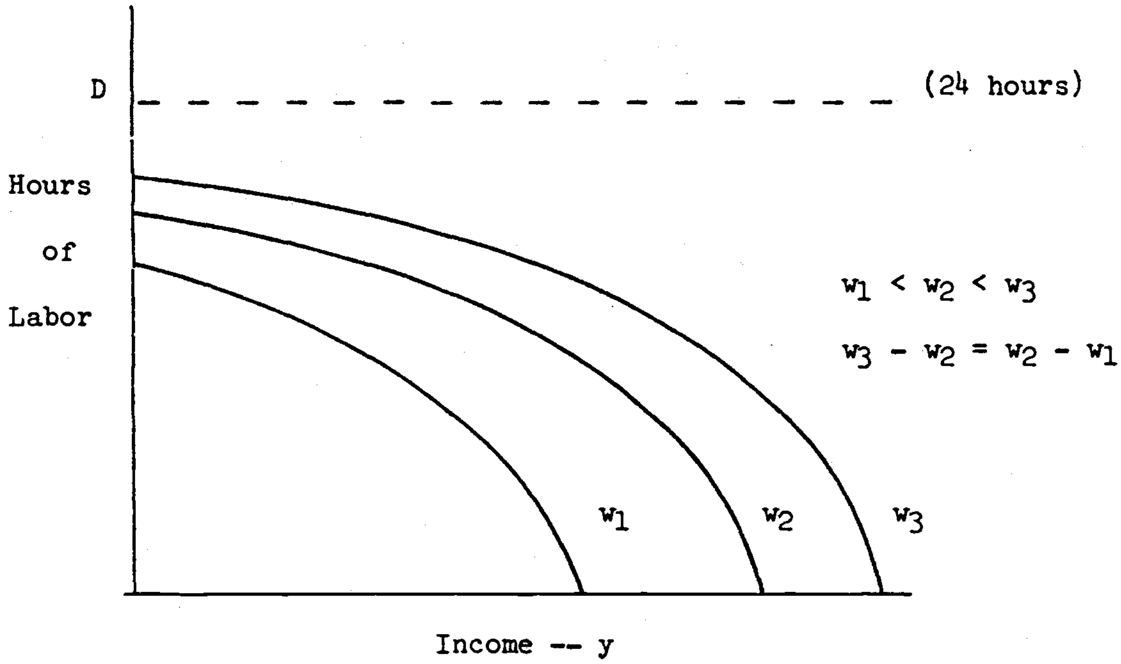


Fig. 1.2: Labor as a function of income, holding wage constant.

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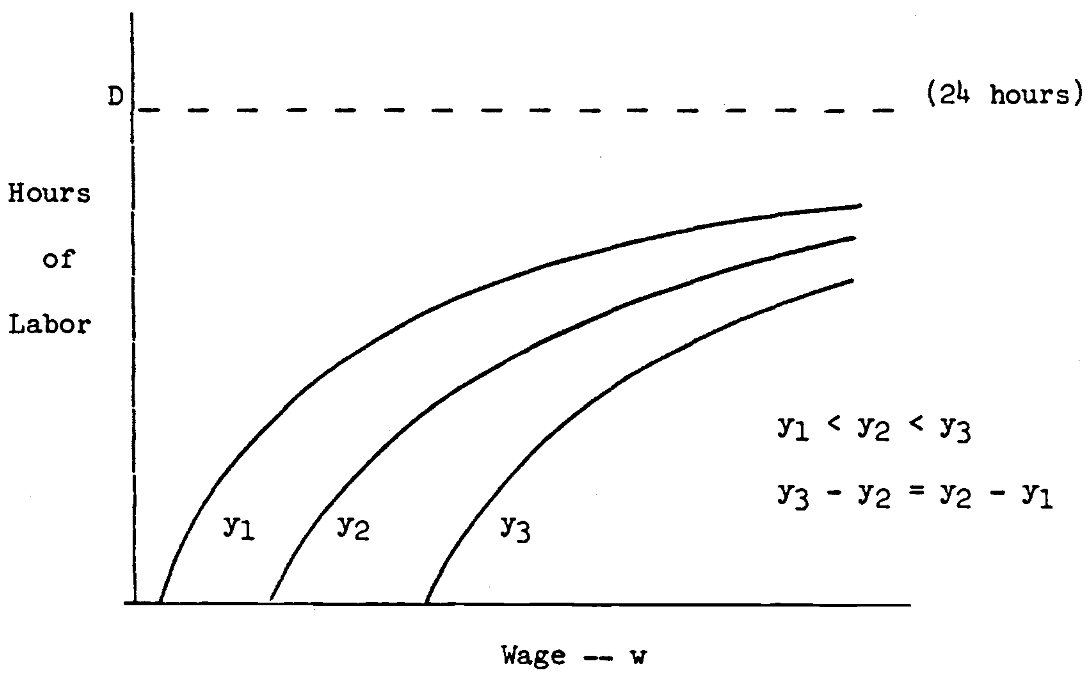

Fig. 1.3: Labor as a function of wage, holding income constant.

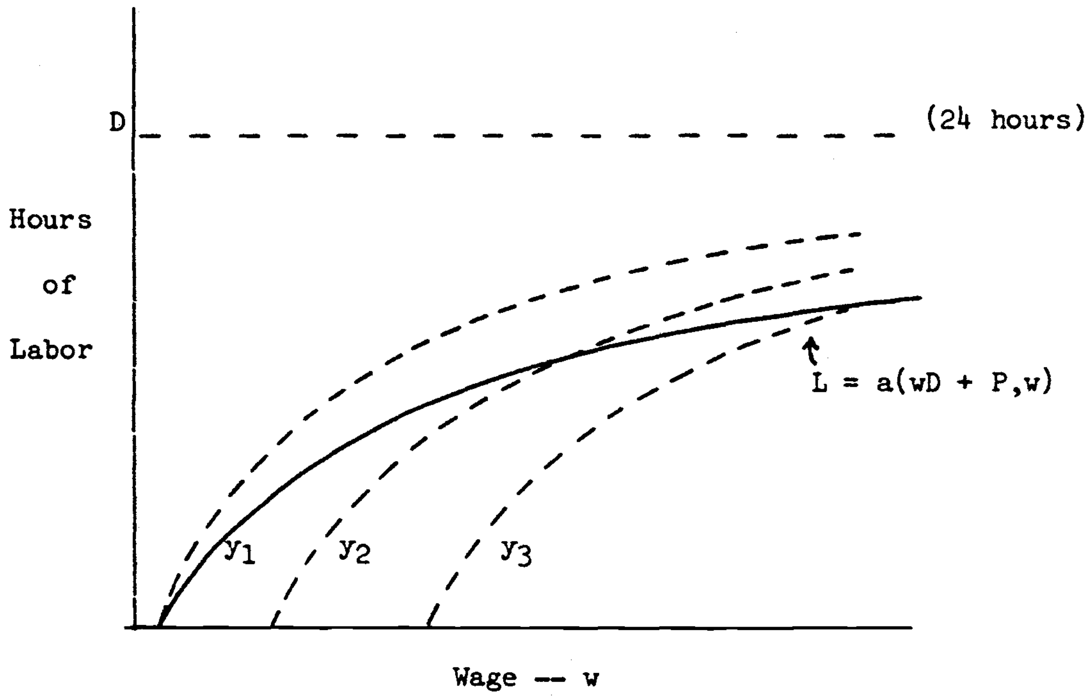

Fig. 1.4: Labor as a function of wage, holding profit constant.

Table 1.3

Consumer-Laborer Model: Effect of Increased Wage or Profit

	$\left.\frac{d}{dw}\right\|_P$		$\left.\frac{d}{dP}\right\|_w \left(= \left.\frac{d}{dy}\right\|_w \right)$	
1. Labor: $L = a(y,w)$	$a_1 D + a_2$	> 0	a_1	< 0
2. Income: $y = P + wD$	D	> 0	1	> 0
3. Consumption: $Q = P + wL$	$L + w \left.\frac{dL}{dw}\right\|_P$	> 0	$1 + wa_1$	> 0 *
4. Utility: $u(Q,Z)$	u_1	> 0	$u_1 L$	> 0

* By the assumption that food is a normal good.

1.5 The Self-Sufficient Farmer[C]

This section combines the labor supply equation developed in Sec. 1.4 with equations for a profit-maximizing land-owning firm. The firm can hire only its owner's labor, as the supervision requirement blocks outside hiring. The exogenous variables of Sec. 1.4, wage and profit, become endogenous functions of one remaining exogenous variable, land size. The partials with respect to land size appear in Table 1.4.

The Maximization Problem:

Suppose the consumer-laborer of 1.4 owns a "firm". The firm in turn owns a piece of land of size T. The firm hires the owner's labor at a given wage, w, and maximizes profits. It returns these profits, P, to the owner. The production function is as described in 1.3.5. There are no explicit transactions costs. However, transactions costs are implicit in the fact that the firm can hire only the owner's labor.

The firm's profit equals output of food, $F = f(T,L)$, less labor costs, wL:

$$(5.1) \qquad P = f(T,L) - wL$$

The firm's output of food, F, necessarily equals the owner's consumption, Q:

$$(5.2) \qquad Q = F = f(T,L)$$

One could also imagine that the firm's owner owned the land, which it rented to the firm at rate r. Then profit would be:

$$(5.3) \qquad P^* = f(T,L) - wL - rT$$

However, I will generally use P and not P* for profit, because it is normal accounting practice to include rent on a firm's assets within profit; it is equally normal practice to exclude wages to the owner. In either case, of course, the owner receives the same lump-sum payment from his firm:

$$(5.4) \qquad P^* + rT = f(T,L) - wL = P$$

When the firm maximizes profit, the first-order conditions are:

$$(5.5) \qquad w - f_2 = 0$$

$$(5.6) \qquad r - f_1 = 0$$

Equation (5.5) implicitly gives the firm's demand for labor as a function of wage and land size. The demand obviously falls as the wage increases, at a given land size. An increase in land size shifts the demand curve outwards, so demand rises as land size increases, holding wage constant.

Equation (5.6) implicitly gives the firm's demand for land as a function of rent and land size.

The Landowner and Firm Combined:

The equations for the firm, (5.1), (5.2), and (5.5) can be combined with those for the income and labor supply of the consumer-laborer: (4.2) and (4.3). The combined equations give labor supply and other variables as a function of land size. Table 1.4 summarizes the effect of an increase in land size on these variables.

Table 1.4

Effect of Increased Land Size, T, on Landowner and Firm

	$\frac{d}{dT}$		
1. Labor: L	$\frac{(a_1 Z + a_2) f_{12} + a_1 f_1}{J}$	> 0 *	see (5.7)
2. Ratio, labor to land: L/T	$\frac{1}{T} [\frac{dL}{dT} - \frac{L}{T}]$	$\leq$ 0	see (5.10)
3. Wage, MP labor: $w = f_2$	$\frac{f_{12} + f_{22} a_1 f_1}{J}$	> 0	see (5.11)
4. Labor cost: LC = wL	$w \frac{dL}{dT} + L \frac{dw}{dT}$	> 0	see (5.12)
5. Labor cost/acre: LC/T = wL/T	$w \frac{d}{dT} (\frac{L}{T}) + \frac{L}{T} \frac{dw}{dT}$	> 0 then < 0	see (5.13)
6. Output: F = f(T,L)	$f_1 + f_2 \frac{dL}{dT}$	> 0	see (5.14)
7. MP land: f_1 = r (rent) = P/T for const retrns	$\frac{f_{11} + a_1 f_1 f_{12} - (a_1 Z + a_2)[f_{11} f_{22} - (f_{12})^2]}{J}$	<0 c.r., d.r. >0 then <0 i.r.	see (5.15)
8. AP labor: F/L	$\frac{1}{L^2} [f_1 L - (F - f_2 L) \frac{dL}{dT}]$	>0 c.r., i.r <0 then >0 d.r.	see (5.16)
9. AP land: F/T	$-\frac{1}{T^2} [F - f_1 T - f_2 T \frac{dL}{dT}]$	<0 c.r., d.r >0 then <0 i.r.	see (5.17)
10. Labor share: LS = wL/F	> 0, by assumption, since $\frac{L}{T}$ falls		see (5.18)
11. Profit: P = F - wL = f_1T for const retrns	$f_1 - L \frac{dw}{dT}$	>0 then <0	see (5.20)
12. Landowner's income: y = P + wD	$\frac{f_{12} Z + (1 - a_2 f_{22}) f_1}{J}$	> 0	
13. Landowner's consptn: Q = F = P + wL	$\frac{dF}{dT}$	> 0	
14. Landowner's utility: U = u(Q,Z)	$u_1 f_1$	> 0	

* $J = 1 - (a_1 Z + a_2) f_{22} > 0$ see (5.7)

Equilibrium Labor Supply and Wage as a Function of Land:

These are the quantity of labor and the wage which equate the owner's labor supply to his firm's labor demand for every value of land size, T.

An increase in land increases equilibrium labor supply by assumption:

$$(5.7) \qquad \frac{dL}{dT} = \frac{(a_1 Z + a_2)f_{12} + a_1 f_1}{1 - (a_1 Z + a_2)f_{22}} > 0$$

The denominator, henceforth denoted as "J", is > 0, because $(a_1 Z + a_2) > (a_1 D + a_2) > 0$ from (4.9).

The entire expression is > 0 by the assumption in 1.3.3h. that a "wage effect" always dominates an "income effect", --so the labor supply curve never bends backwards.

It must also be true, by the assumption that L tends asymptotically towards a maximum, that:

$$(5.8) \qquad \frac{d^2L}{dT^2} < 0$$

$$(5.9) \qquad \frac{dL}{dT} \longrightarrow 0 \quad \text{as T gets large}$$

In addition, L mst ---> 0 as T ---> 0, although the ratio L/T will reach a finite maximum, by the assumption 1.3.5e. that f_2 ---> 0 when the ratio gets high enough.

Figures 1.5 and 1.6 show L and L/T as a function of T. L is concave from the origin with dL/dT = maximum L/T at T = 0. L/T declines from its maximum value as T increases:

$$(5.10) \qquad \frac{d}{dT}\left(\frac{L}{T}\right) = \frac{1}{T}\left[\frac{dL}{dT} - \frac{L}{T}\right] \leq 0 \quad \text{(= 0 only for T = 0, L/T at a max.)}$$

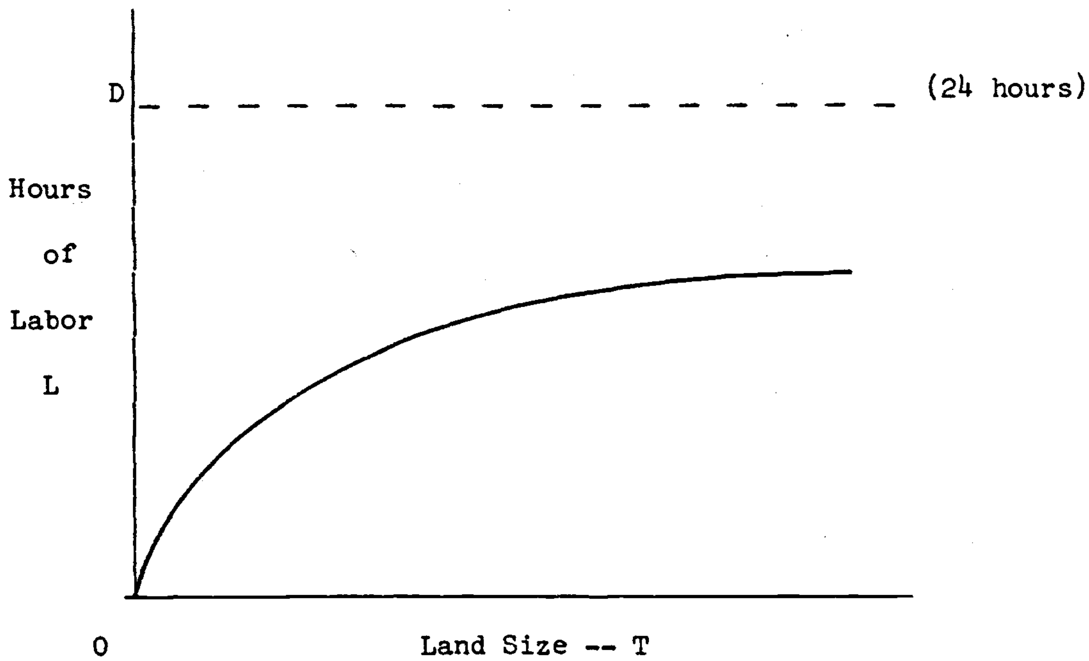

Fig. 1.5: Labor as a function of land size: L(T).

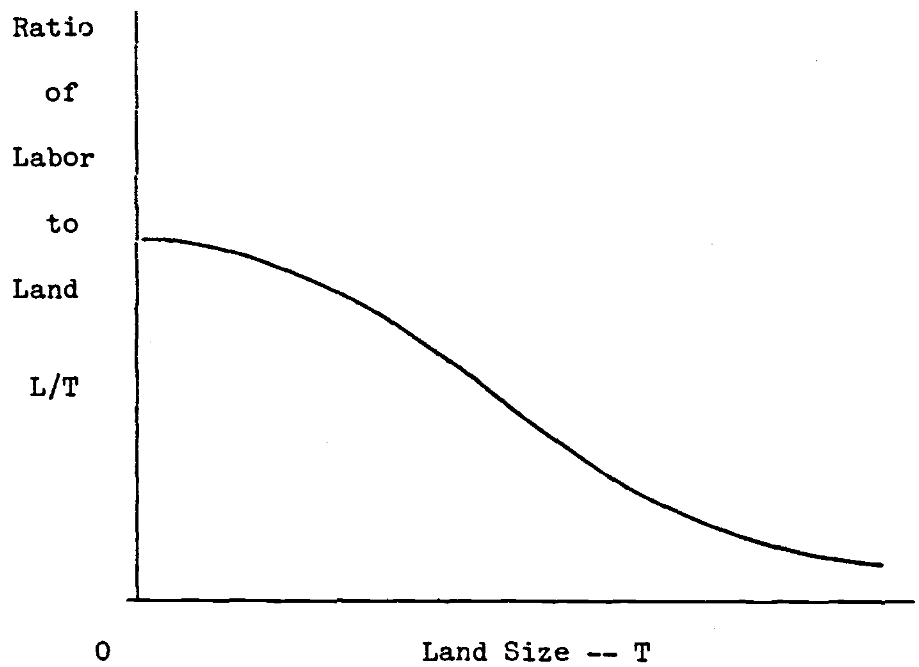

Fig. 1.6: Ratio of labor to land as a function of land size.

The wage also implicitly depends on land size, T. By (5.5), the wage everywhere equals $f_2(T,L)$. Call the latter the "net marginal product of labor", because L is a function of T, to distinguish it from the "ordinary marginal product of labor", $f_2(T,L_o)$, where L_o is any constant quantity of labor.

The effect of an increase in land size on the wage and net marginal product of labor is:

$$(5.11) \quad \frac{dw}{dT} = \frac{d(f_2)}{dT} = f_{12} + f_{22}\frac{dL}{dT}$$

$$= \frac{f_{12} + f_{22}a_1f_1}{J} > 0$$

By assumption, w and $f_2(T,L)$ start at 0 when T = 0, and L/T is at or near a maximum. w and f_2 rise toward a maximum as L/T falls and the marginal product of land, f_1, approaches 0.

Figure 1.7 shows w and $f_2(T,L)$ as a function of land size. The function is S-shaped, because for small values of T, the ratio L/T remains at its maximum (due to assumption 5e.) where $f_2 = 0$. For comparison, Fig. 1.7 also shows the "ordinary" marginal product of labor, $f_2(T,L_o)$ as a function of land size.

Labor Cost and Labor Cost per Acre:

Since both wage and labor increase with land size, obviously so does labor cost, wL.

$$(5.12) \quad \frac{d}{dT}(wL) > 0$$

And since the wage rises in an S, so must labor cost. Labor cost as a function of land size appears in Fig. 1.9.

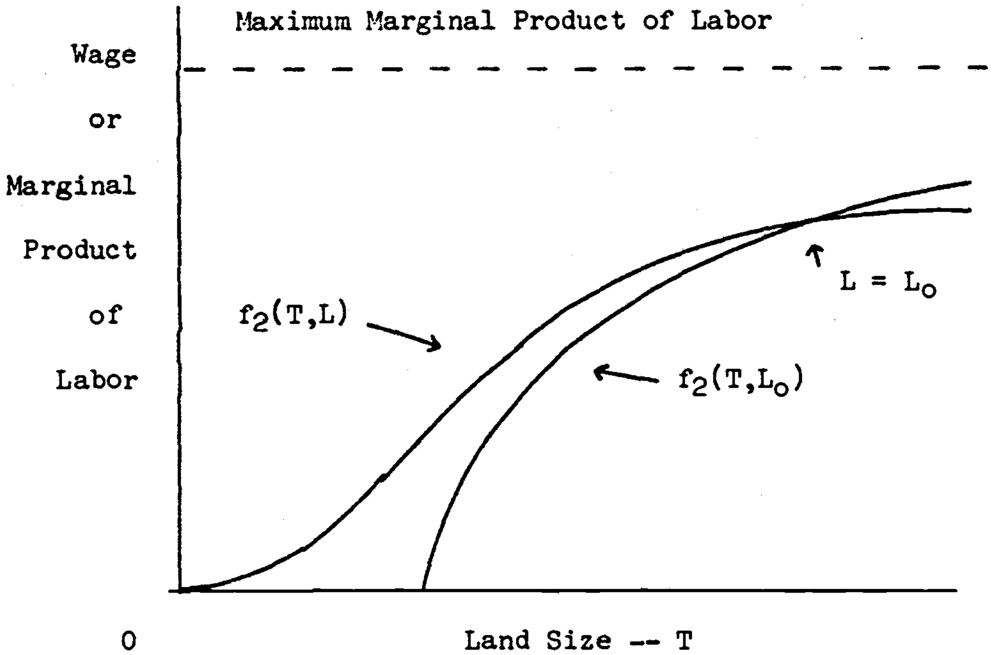

Fig. 1.7: Wage and net marginal product of labor as a function of land size: $w = f_2(T,L)$. Ordinary marginal product of labor as a function of land size: $f_2(T,L_o)$, where L_o is an arbitrary constant quantity of labor.

. . . .

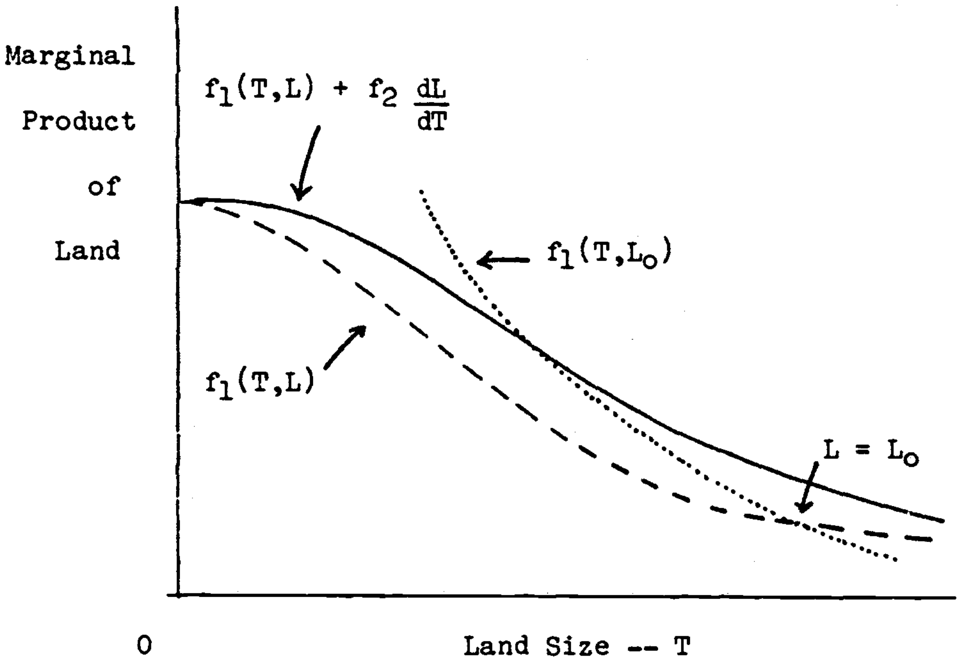

Fig. 1.8: Marginal physical product of land $[f_1(T,L) + f_2(T,L) \frac{dL}{dT}]$, net marginal product of land $[f_1(T,L)]$, and ordinary marginal product of land $[f_1(T,L_o)]$, -- all as a function of land size.

Labor cost per acre rises and then falls again:

$$(5.13)\qquad \frac{d}{dT}\left(\frac{wL}{T}\right) = w\frac{d}{dT}\left(\frac{L}{T}\right) + \frac{L}{T}\frac{dw}{dT}$$

$$= \frac{L}{T}\frac{dw}{dT} > 0 \qquad \text{small } T \text{ (where } w \to 0)$$

$$= w\frac{d}{dT}\left(\frac{L}{T}\right) < 0 \qquad \text{large } T \text{ (where } \frac{dw}{dT} \to 0)$$

Output, Marginal Product of Land and Rent:

Output obviously increases as land size increases:

$$(5.14)\qquad \frac{dF}{dT} = f_1 + f_2\frac{dL}{dT}$$

Call dF/dT the "marginal physical product of land", to distinguish it from the "net marginal product of land": $f_1(T,L)$. This should in turn be distinguished from the "ordinary marginal product of land", $f_1(T,L_o)$ where L_o is some constant quantity of labor.

All three appear in Figure 1.8.

(One might still wonder, despite the mathematics, why rent equals $f_1(T,L)$, and not $\frac{d}{dT}f(T,L)$. The answer is that the real output of the landowner's firm is food, $F = f(T,L)$, plus the landowner's leisure, wZ. And so the real net marginal product of an increment of land is:

$$\frac{d}{dT}[f(T,L) + wZ] = f_1 + (f_2 - w)\frac{dL}{dT} = f_1$$

That is the reason for calling $\frac{d}{dT}f(T,L)$ the "marginal physical product of land".)

The rent and net marginal product of land also implicitly depend on land size, so that:

$$(5.15) \quad \frac{dr}{dT} = \frac{df_1}{dT} = f_{11} + f_{12}\frac{dL}{dT}$$

$$= \frac{f_{11} + a_1 f_1 f_{12} - (a_1 Z + a_2)[f_{11} f_{22} - (f_{12})^2]}{J}$$

This is obviously < 0 for constant or diminishing returns to scale. It may be > 0 for increasing returns and small values of T, but must become < 0 for larger values of T, where transactions costs ensure net diseconomies of scale.

Fig. 1.9 shows output as a function of land size, together with labor cost. Assuming net marginal product of land, $f_1(T,L)$, converges asymptotically to 0 as T increases, then output must converge to a maximum. However, it does not converge as fast as labor, so that average product of labor continues to increase (see below).

Average Product of Labor and Land:

An increase in land affects the average physical product of labor:

$$(5.16) \quad \frac{d}{dT}\frac{[f(T,L)]}{L} = \frac{f_1 L - (f - f_2 L)\frac{dL}{dT}}{L^2}$$

$$= \frac{f_1}{L} > 0 \qquad \text{for large T}$$

$$= \frac{1}{LT}(f_1 T - f) \qquad \text{for small T}$$

(since for $T = 0$, $f_2 = 0$ and $dL/dT = L/T$)

So the average product of labor always increases for large T. It also increases for small T for constant or increasing returns to scale, so that $f \leq f_1 T + f_2 L$. In the wildly implausible case of decreasing returns at the smallest scale, it may fall a bit before rising.

An increase in land affects the average product of land:

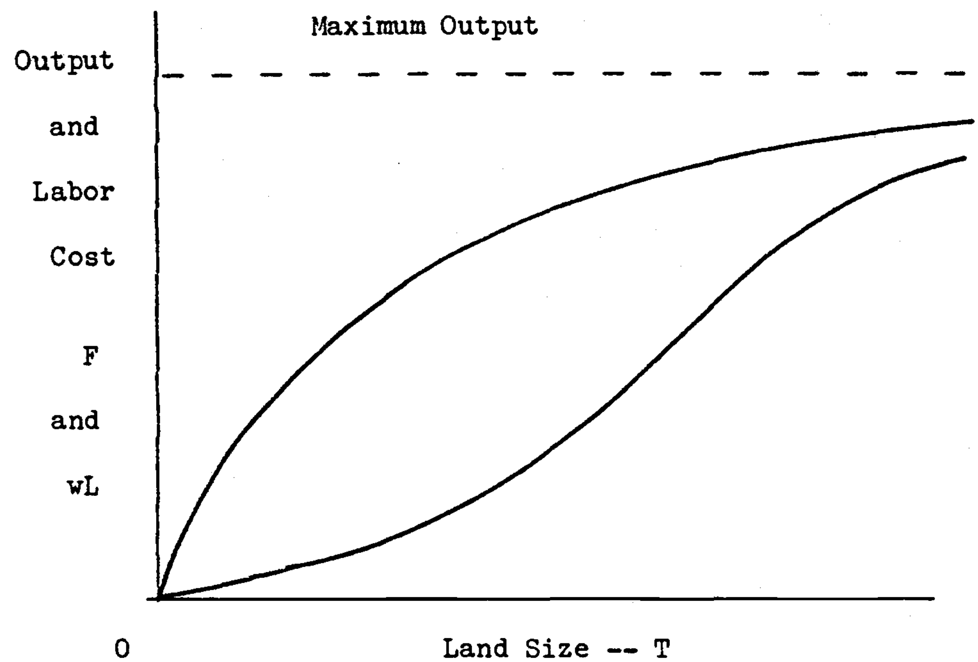

Fig. 1.9: Output as a function of land size: $F = f(T,L)$. Labor cost as a function of land size: $wL = f_2L$.

. . . .

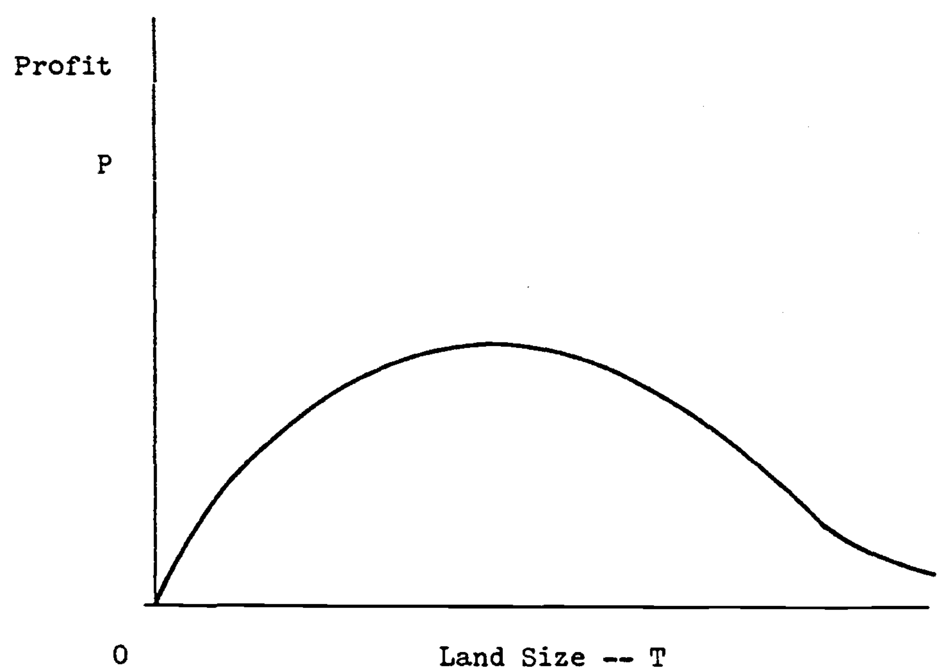

Fig. 1.10: Profit as a function of land size: $P = f(T,L) - wL$.

$$(5.17)\quad \frac{d}{dT}\frac{f(T,L)}{T} = -\frac{1}{T^2}\left[f - f_1 T - f_2 T \frac{dL}{dT} \right]$$

$$= -\frac{1}{T^2}\left[f - f_1 T \right] < 0 \qquad \text{for large } T \text{ (where } f_1 \text{ gets very small)}$$

$$= -\frac{1}{T^2}\left[f - f_1 T \right] \qquad \text{for small } T \text{ (where } f_2 \to 0)$$

For constant or decreasing returns, the average physical product of land falls everywhere. For increasing returns, it rises for small T, where f_2 close to 0, so that $f < f_1 T$. But it falls again for larger T, where transactions costs ensure net diseconomies of scale.

Labor Share of Output; Profit Share; Rent Share:

The labor share of output, LS, is just labor cost, $wL = f_2 L$, divided by output f(T,L). By assumption 1.3.5e, labor share rises as the ratio of labor to land falls.

$$(5.18)\quad \frac{d(LS)}{dT} = \frac{d}{dT}\left[\frac{wL}{f(T,L)}\right] > 0$$

Since profit share, PS, equals 1 - LS, then necessarily:

$$(5.19)\quad \frac{d(PS)}{dT} < 0$$

Rent share, RS, is the payment to land, $rT = f_1 T$, divided by output f(T,L). By the assumption in 1.3.5c that $f_1 T$ falls as the ratio of labor to land falls, RS also falls.

By (5.4):

$$1 = LS + RS + \frac{P^*}{f(T,L)}$$

where $P^* = 0$ for constant returns to scale, $P^* > 0$ for increasing returns, and $P^* < 0$ for decreasing returns. So for constant returns, RS = PS. For increasing returns, RS > PS, and for decreasing returns, RS < PS.

Given increasing returns, it is therefore possible that RS > 1 where LS is close to 0 for very small land; and that LS > 1 where RS is close to 0 for very large land. This result may seem bizarre. But actually it means nothing. For the landowner receives the output of his firm in a lump, and doesn't care about the breakdown into shares.

Profit and Profit per Acre:

Profit $P = f(T,L) - wL = f_1T + P^*$. So for constant returns to scale, profit $P = f_1T$. Profit rises for small land size:

$$\text{(5.20)} \quad \frac{dP}{dT} = f_1 - L\frac{dw}{dT} \longrightarrow 0 \quad \text{large } T$$

$$= f_1 > 0 \quad \text{small } T \text{ (where } L,T \longrightarrow 0)$$

It necessarily falls again, by the assumption that the labor share of output rises as the ratio of land to labor falls.

Profit/acre $P/T = f_1 + P^*/T$. Profit per acre presumably behaves like the net marginal product of land, f_1 -- necessarily so for constant returns to scale so that $P^* = 0$.

Profit appears in Figure 1.10.

1.6 The Peasant: The Landowner Who Works for Hire[c]

This section combines the labor supply equation from Sec. 1.4 with a profit-maximizing firm, under the assumption that the firm owner, "the peasant" also works freely at a given market wage, v.

This model so far is identical to that of the preceding section 1.5 except for boundary conditions.

For suppose the firm hires all the peasant's labor, L, at his wage, w, applies A of it to the land, and hires out H of it at the exogenous wage, v. So the peasant's labor supply is:

(6.1) $L = A + H$

And the firm maximizes its profit

(6.2) $P = f(T,A) + vH - wL$

subject to 6.1 and the requirement that hired labor $H \geq 0$. This produces Kuhn-Tucker conditions:

(6.3) $w - f_2 = 0$

(6.4) $w - v \geq 0 \qquad (w - v)H = 0$

The first condition states, like 5.5, that peasant's wage, w, equals marginal product of labor, f_2. The second condition states that the firm will not hire out extra labor, H, as long as the owner's wage $w = f_2$ exceeds the external wage v, at $H = 0$. So if $w > v$--as must happen if land size, T, is large enough, or external wage, v, is low enough--then $H = 0$ --and the model is identical to the model in Section 1.5.

If equality holds in the second condition, so $w = v$, then the marginal product of labor f_2 must equal the external wage, regardless of the size of the peasant's landholding. So instead of one exogenous variable as in Sec. 1.5, land size T; there are now two: T and external wage, v.

Only his wealth, and thus the size of his profit, P, affects the peasant's behavior. So the richer the peasant, the less he works. Table 1.5 summarizes this and other results, as functions of the exogeonous variables, T and v, derived from combining equations 6.1 through 6.4 with the labor supply equation, 4.3: L = a(y,w). These results will be used in the general equilibrium model in Chapter 7.

A Truly Transaction Cost-less Model:

The model as presented so far does implicitly include transactions costs in the assumption that rental of land isn't permitted. Suppose we permit rental, and, to keep everything from going haywire, necessarily assume a linear homogeneous production function. So rental rate, r, which equals the marginal product of land, f_1, is necessarily fixed by external wage v.

Making these further assumptions, the only significant effect is to cut the peasant fully free from his firm. He receives profit $P = f_1 T$ from it, but the applied labor A on it could be anybody's.

So Table 1.5, which assumes a linear homogeneous production function, also summarizes the results for the fully transaction cost less world.

Table 1.5

Effect of an Increase in Land, T, or Wage, v.

	$\left.\frac{d}{dT}\right\|_v$	$\left.\frac{d}{dv}\right\|_T$
1. Labor: Landowner's total: L	$a_1 f_1 < 0$	$a_1(D - A) + a_2 > 0$
Self = applied: S = A	$-\frac{f_{12}}{f_{22}} > 0$	$\frac{1}{f_{22}} < 0$
Hired: H	$a_1 f_1 + \frac{f_{12}}{f_{22}} < 0$	$a_1(D-A) + a_2 - \frac{1}{f_{22}} > 0$
2. Ratio, labor to land: A/T	0	$\frac{1}{T}\frac{dA}{dv} < 0$
3. Wage, MP labor: $w = f_2 = v$	0	$1 > 0$
4. Labor cost: LC = vA	$v\frac{dA}{dT} > 0$	$A + v\frac{dA}{dv}$?
5. Labor cost/acre: LC/T = vA/T	0	$\frac{1}{T}\frac{dLC}{dv}$?
6. Output: F = f(T,A)	$f_1 + v\frac{dA}{dT} > 0$	$v\frac{dA}{dv} < 0$
7. MP land: f_1 = Profit/acre	$f_{11} - \frac{(f_{12})^2}{f_{22}} = 0$	$f_{12}\frac{dA}{dv} < 0$
8. AP labor: F/A	0	$-\frac{F - vA}{A^2}\frac{dA}{dv} > 0$
9. AP land: output/acre = F/T	0	$\frac{1}{T}\frac{dF}{dv} < 0$
10. Labor share: LS = vA/F	0	> 0 by assumption because $\frac{A}{T}$ falls
11. Profit: $P = F - vA = f_1 T$	$f_1 > 0$	$-A < 0$
12. Landowner's income: y = P + vD	$f_1 > 0$	$D - A > 0$
13. Landowner's consptn: Q = P + vL	$f_1(1 + va_1) > 0$	$H + v\frac{dL}{dv} > 0$
14. Landowner's utility: U = u(Q,Z)	$u_1 f_1 > 0$	$u_1 H > 0$

1.7 The Small Landlord: the Landowner Who Works Himself on his Land, and Hires and Supervises Additional Labor

The small landlord is the same model as that in Sec. 1.5 and 1.6, but with yet another set of boundary conditions, and with transactions costs appearing explicitly.

The small landlord can hire additional labor, H, at the given wage, v. But he must supervise this labor at rate k: $0 \leq k \leq 1$. He also applies S hours of his own labor directly to his land. So total labor applied directly to the land is:

(7.1) $$A = S + H$$

And the landowner's total labor is:

(7.2) $$L = S + kH$$

The landlord's firm hires both his labor L, and hired labor H, and maximizes its profit:

(7.2) $$P = f(T,A) - wL - vH$$

subject to 7.1 and 7.2, plus $S,H \geq 0$. This gets the Kuhn-Tucker conditions:

(7.3) $$w - f_2 \geq 0 \qquad (w - f_2)S = 0$$

(7.4) $$v + kw - f_2 \geq 0 \qquad (v + kw - f_2)H = 0$$

Condition (7.3) states that if the landowner works directly on his own land, ie., $S > 0$, then the marginal product of his direct labor equals his wage. He will not work directly on his land -- $S = 0$ -- if his wage exceeds the marginal product of his direct labor when $S = 0$. This will be the case for the large landlord in Section 1.8, following.

Condition (7.4) states that the firm will hire outside labor, ie. $H > 0$, only if the employee's wage, v, plus k times the landowner's wage for supervision, kw, equals the marginal product of labor, f_2. That is, labor cost equals marginal product. If the cost exceeds marginal product, the firm hires no outside labor. If the equality holds for condition (7.3), but not for (7.4)--which must be the case if k is large enough--then $H = 0$ and the model becomes identical to the self-sufficient farmer in 1.5.

If the equality holds for both conditions, then:

$$(7.5) \qquad w = f_2 = \frac{v}{1 - k}$$

which are the conditions for the small landlord.

Note that (7.5) means that for $k > 0$, the marginal product of labor and the landlord's wage exceed the wage paid the employee(s).

If $k = 0$ and there is no constraint $H \geq 0$, then the small landlord model becomes continuous with the peasant model in Section 1.6. $H > 0$ when the landowner hires labor, and $H < 0$ when he works for hire, all at given wage, v.

Equations (7.1) through (7.5) can be combined with the consumer-laborer equations (4.2) and (4.3) to obtain all variables as a function of three exogenous variables: land size, T; external wage, v; and supervision rate, k. These results appear in Table 1.6, which also assumes a linear homogeneous production function, for simplicity and ease of comparison with the peasant in Table 1.5.

In many ways, the small landlord more closely resembles the peasant of Sec. 1.6 than the self-sufficient farmer of 1.5. For marginal product of labor remains fixed by the market wage, v. This happens because the landowner does work identical to that of his employee(s), making his wage necessarily proportional to his employees' wage, v. Moreover, when T increases, the weighted average wage falls from $v/(1 - k)$ when $H = 0$, to $v/(1 + k)$ when $S = 0$. So, including supervisory labor, the marginal and average products of labor fall as land size increases. Nevertheless, the small landlord's wage remains higher than his employees', and ratio of labor to land remains lower, and the average product of labor remains higher than in the absence of a supervision requirement.

Table 1.6

Effect of an Increase in Land, T, Wage, v, or Supervision Rate, k.

	$\frac{d}{dT}\Big\|_{v,k}$	$\frac{d}{dv}\Big\|_{T,k}$	$\frac{d}{dk}\Big\|_{T,v}$
1. Labor:			
Landowner's total: L L = S + kH	$a_1 f_1 \quad < 0$	$\frac{a_1(D-A) + a_2}{1 - k} \quad > 0$	$\frac{v[a_1(D-A) + a_2]}{(1-k)^2} \quad > 0$
Landowner's self: S	$\frac{a_1 f_1}{1-k} + \frac{k}{1-k}\frac{f_{12}}{f_{22}} \quad < 0$	$\frac{1}{(1-k)^2}\left[a_1(D-A)+a_2 - \frac{k}{f_{22}}\right] \quad > 0$	$\frac{v}{(1-k)^3}\left[a_1(D-A)+a_2 - \frac{k}{f_{22}}\right] - \frac{H}{1-k} \quad >0$* mostly
Hired. H	$-\frac{f_{12}}{(1-k)f_{22}} - \frac{a_1 f_1}{1-k} \quad > 0$	$\frac{-1}{(1-k)^2}\left[a_1(D-A)+a_2 - \frac{1}{f_{22}}\right] \quad < 0$	$\frac{-v}{(1-k)^3}\left[a_1(D-A)+a_2 - \frac{1}{f_{22}}\right] + \frac{H}{1-k} \quad <0$* mostly
Total applied: A = S + H	$-\frac{f_{12}}{f_{22}} \quad > 0$	$\frac{1}{(1-k)f_{22}} \quad < 0$	$\frac{v}{(1-k)^2 f_{22}} \quad < 0$
Grand total: L + H = A + kH	$\frac{dA}{dT} + k\frac{dH}{dT} \quad > 0$	$\frac{dA}{dv} + k\frac{dH}{dv} \quad < 0$	$\frac{dA}{dk} + H + k\frac{dH}{dk} \quad < 0$ * mostly
2. Ratio, labor to land:			
Applied: A/T	0	$\frac{1}{T}\frac{dA}{dv} \quad < 0$	$\frac{1}{T}\frac{dA}{dk} \quad < 0$
Total: $\frac{A + kH}{T}$	> 0, frm $\frac{A}{T}$ to $\frac{(1+k)A}{T}$	$\frac{1}{T}\left[\frac{dA}{dv} + k\frac{dH}{dv}\right] \quad < 0$	$\frac{1}{T}\left[\frac{dA}{dk} + H + k\frac{dH}{dk}\right] \quad < 0$* mostly

Table 1.6, continued

	$\frac{d}{dT}\Big\|_{v,k}$	$\frac{d}{dv}\Big\|_{T,k}$	$\frac{d}{dk}\Big\|_{T,v}$
3. Wage and MP labor:			
Employee's: v	0	$1 > 0$	0
$w = f_2 = \frac{v}{1-k}$	0	$\frac{1}{1-k} > 0$	$\frac{v}{(1-k)^2} > 0$
Weighted av $\frac{wL + vH}{L + H}$	< 0, frm $\frac{v}{1-k}$ to $\frac{v}{1+k}$	> 0, frm $\frac{v^a}{1+k}$ to $\frac{v^b}{1-k}$, $v^a < v^b$	> 0, frm v to $\frac{v}{1-k}$
4. Labor cost. $LC = wL + vH = f_2A$	$f_2\frac{dA}{dT} > 0$	$\frac{A}{1-k} + f_2\frac{dA}{dv}$?	$A\frac{df_2}{dk} + f_2\frac{dA}{dk}$?
5. Labor cost/acre. $LC/T = f_2A/T$	0	$\frac{1}{T}\frac{dLC}{dv}$	$\frac{1}{T}\frac{dLC}{dk}$
6. Output: $F = f(T,A)$	$f_1 + v\frac{dA}{dT} > 0$	$f_2\frac{dA}{dv} < 0$	$f_2\frac{dA}{dk} < 0$
7. MP land: $f_1 = P/T$ (const retrns)	$f_{11} - \frac{(f_{12})^2}{f_{22}} = 0$ (c.r.)	$f_{12}\frac{dA}{dv} < 0$	$f_{12}\frac{dA}{dk} < 0$
8. AP labor			
F/A	0	$-\frac{F - wA}{A^2}\frac{dA}{dv} > 0$	$-\frac{F - wA}{A^2}\frac{dA}{dk} > 0$
$\frac{F}{A + kH} = \frac{F}{L + H}$	< 0, frm $\frac{F}{A}$ to $\frac{F}{A(1+k)}$	> 0, frm $\frac{1}{1+k}\frac{F^a}{A}$ to $\frac{F^b}{A}$, $a<b$	> 0, frm $\frac{F^a}{A}$ to $\frac{F^b}{A}$ $a<b$

Table 1.6, continued

	$\frac{d}{dT}\Big\|_{v,k}$	$\frac{d}{dv}\Big\|_{T,k}$	$\frac{d}{dk}\Big\|_{T,v}$
9. AP land: = output/acre F/T	0	$\frac{1}{T}\frac{dF}{dv} < 0$	$\frac{1}{T}\frac{dF}{dk} < 0$
10. Labor share: $LS = f_2A/F = wA/F$	0	> 0, because $\frac{A}{T}$ falls	> 0, because $\frac{A}{T}$ falls
11. Profit: $P = F - wL - vH = F - f_2A$	$f_1 > 0$	$-\frac{A}{1-k} < 0$	$-\frac{wA}{1-k} < 0$
12. Landowner's income: $y = P + wD$	$f_1 > 0$	$\frac{D-A}{1-k} > 0$ if $D > A$	$\frac{w(D-A)}{1-k} > 0$ if $D > A$
13. Landowner's consptn: $Q = P + wL$	$f_1(1 + wa_1)$	$w\frac{dL}{dv} - H > 0$ small H, else ?	$w\left[\frac{dL}{dk} - H\right] > 0$ small H, else ?
14. Landowner's utility: $U = u(Q,Z)$	$u_1f_1 > 0$	$-u_1H < 0$	$-u_1wH < 0$

* For small H, as k increases, $\frac{dS}{dk} > 0$, $\frac{dH}{dk} < 0$ and $\frac{d}{dk}(A + kH) < 0$, as would be expected. However, it is possible for very small k that S will fall a bit at first, and H rise a bit at first, as the landowner shifts labor from the land to supervising employees. As k increases further, S must rise and H must fall.

1.8 The Large Landlord: The Landowner Who Only Supervises Hired Labor

Suppose the inequality holds in the first of the Kuhn-Tucker conditions in Section 1.7, (7.3), --as it must if a landowner owns enough land, and $k < 1$. Then:

(8.1) $$w - f_2 \geq 0 \qquad \text{and} \qquad S = 0$$

with the equality holding only at the boundary where S --> 0.

So the landowner does not work directly on the land, because his wage exceeds the marginal product of his labor there. Then, assuming the equality holds for the second Kuhn-Tucker condition, (7.4),

(8.2) $$v + kw - f_2 = 0 \quad \text{and} \quad H \geq 0$$

it follows that:

(8.3) $$w \geq f_2 \geq \frac{v}{1-k}$$

with equality only at the boundary where S --> 0. That is, the large landlord's personal wage exceeds the marginal product of labor on his land, which in turn exceeds $v/(1-k)$.

In the special case where $k = 0$:

(8.4) $$w \geq f_2 = v$$

with $w = f_2$ only at the boundary where S --> 0.

From the second condition it also must also follow that:

(8.5) $$w = \frac{f_2 - v}{k}$$

except in the special case where $k = 0$ and (8.4) holds.

<u>The Large Landlord is Equivalent to the Self-Sufficient Farmer:</u>

To show that the large landlord is mathematically identical to the self-sufficient farmer of Sec. 1.5, define:

$$(8.6) \qquad g(T,L) \;\underline{=}\; f(T,L/k) - vL/k \;=\; f(T,H) - vH$$

g(T,L) also implicitly depends on v and k. It is the firm's output net of hired labor cost, and so equals the owner's consumption, Q:

$$(8.7) \qquad g(T,L) = 0$$

It has the same properties as the ordinary production function:

$$(8.8) \qquad g_2 = \frac{f_2 - v}{k} = w > 0$$

$$(8.9) \qquad g_1 = f_1 > 0$$

$$(8.10) \qquad g_{12} = \frac{f_{12}}{k} > 0$$

$$(8.11) \qquad g_{11} = f_{11} < 0$$

$$(8.12) \qquad g_{22} = \frac{1}{k^2} f_{22} < 0$$

$$(8.13) \qquad g_{11}g_{22} - (g_{12})^2 = \frac{1}{k^2}[f_{11}f_{22} - (f_{12})^2]$$

Consequently, g(T,L) as a function of T behaves exactly like f(T,L) as a function of T in Section 1.5 --a useful fact in determining the sign of some of the derivatives with respect to T in the table of derivatives, Table 1.7, following. g(T,L) also simplifies the notation in many cases.

Table 1.7

Effect of an Increase in Land, T, Wage, v, or Supervision Rate, k.

	$\frac{d}{dT}\Big\vert_{v,k}$	$\frac{d}{dv}\Big\vert_{T,k}$	$\frac{d}{dk}\Big\vert_{T,v}$
1. Labor:			
Landowner's total: L $L = kH$	$k\frac{dH}{dT} > 0$	$k\frac{dH}{dv} < 0$	$H + k\frac{dH}{dk} > 0$ then < 0 ?**
Hired = applied: $H = A$	$\frac{(a_1Z+a_2)g_{12} + a_1g_1}{kJ} > 0$ *	$\frac{a_1D + a_2}{k^2J} < 0$	$-\frac{[g_2(a_1D+a_2) + L]}{k^2J} < 0$
Grand total: $L + H$ $= (1+k)H$	$(1+k)\frac{dH}{dT} > 0$	$(1+k)\frac{dH}{dv} < 0$	$H + (1+k)\frac{dH}{dk}$?
2. Ratio, labor to land:			
Applied: $A/T = H/T$	$\frac{1}{T}\left[\frac{dH}{dT} - \frac{H}{T}\right] < 0$	$\frac{1}{T}\frac{dH}{dv} < 0$	$\frac{1}{T}\frac{dH}{dk} < 0$
Total: $\frac{(1+k)H}{T}$	$\frac{(1+k)}{T}\left[\frac{dH}{dT} - \frac{H}{T}\right] < 0$	$\frac{1+k}{T}\frac{dH}{dv} < 0$	$\frac{1}{T}\left[H + (1+k)\frac{dH}{dk}\right]$?

* $J = 1 - g_{22}(a_1Z + a_2) = 1 - \frac{1}{k^2}f_{22}(a_1Z + a_2) > 0$

** $\frac{dL}{dk} = -\frac{[(f_2-v)(a_1D+a_2) + f_{22}H(a_1Z+a_2)]}{k^2J}$. At $k = 0$, $\frac{dL}{dk} = H > 0$. By assumption, L increases at least as long as the wage increases, particularly if income also falls.

Table 1.7, continued

	$\frac{d}{dT}\Big\vert_{v,k}$	$\frac{d}{dv}\Big\vert_{T,k}$	$\frac{d}{dk}\Big\vert_{T,v}$
3. Wage and MP labor:			
Employee's: v	0	$1 \quad > 0$	0
$w = g_2 = \frac{f_2 - v}{k}$	$\frac{g_{12} + g_{22}a_1g_1}{J} \quad > 0$	$-\frac{(1 + g_{22}a_1L)}{kJ} \quad < 0$	$-\frac{[g_2+g_{22}L(1+a_1w)]}{kJ}$ >0 then <0?*
MP applied: f_2	$k\frac{dg_2}{dT} \quad > 0$	$f_{22}\frac{dH}{dv} \quad > 0$	$f_{22}\frac{dH}{dk} \quad > 0$
Wghtd av: $\frac{wL+vH}{L + H} = \frac{f_2}{1+k}$	$\frac{k}{1+k}\frac{dg_2}{dT} \quad > 0$	$\frac{1}{1+k}f_{22}\frac{dH}{dv} \quad > 0$	$\frac{1}{1+k}[f_{22}\frac{dH}{dk} - \frac{f_2}{1+k}]$?
4. Labor cost: $LC = wL + vH = f_2H$	$f_2\frac{dH}{dT} + H\frac{df_2}{dT} \quad > 0$	$(f_{22}H + f_2)\frac{dH}{dv}$?	$(f_{22}H + f_2)\frac{dH}{dk}$?
5. Labor cost/acre: $LC/T = f_2H/T$	> 0 (?) then < 0	$\frac{1}{T}\frac{dLC}{dv}$?	$\frac{1}{T}\frac{dLC}{dk}$?
6. Output: $F = f(T,H)$	$f_1 + f_2\frac{dH}{dT} \quad > 0$	$f_2\frac{dH}{dv} \quad < 0$	$f_2\frac{dH}{dk} \quad < 0$
7. MP land: $f_1 = g_1 = r = P/T$ (const retrns)	$\frac{g_{11} + a_1g_1g_{12}}{J} \quad < 0$	$f_{12}\frac{dH}{dv} \quad < 0$	$f_{12}\frac{dH}{dk} \quad < 0$

* $\frac{dw}{dk} = -\frac{[f_2-v + f_{22}H(1+a_1w)]}{k^2 + f_{22}(a_1Z+a_2)}$. At $k = 0$, $\frac{dw}{dk} = \frac{H(1+a_1w)}{a_1Z + a_2} > 0$. It is apparent from inspection that $d^2w/dk^2 < 0$. So as k increases from 0, the wage rises at a decreasing rate, and may even fall again before w and f_2 converge to $v/(1-k)$, and S becomes > 0, so that the model of Section 7 applies.

Table 1.7, continued

	$\frac{d}{dT}\Big\vert_{v,k}$	$\frac{d}{dv}\Big\vert_{T,k}$	$\frac{d}{dk}\Big\vert_{T,v}$
8. AP labor:			
F/H	$\frac{1}{H^2}[f_1H - (F-f_2H)\frac{dH}{dT}] > 0$	$-\frac{(F-f_2H)}{H^2}\frac{dH}{dv} > 0$	$-\frac{(F-f_2H)}{H^2}\frac{dH}{dk} > 0$
$\frac{F}{(1+k)H} = \frac{F}{L+H}$	$\frac{1}{1+k}\frac{d}{dT}(\frac{F}{H}) > 0$	$\frac{1}{1+k}\frac{d}{dv}(\frac{F}{H}) > 0$	$-\frac{(F-f_2H)}{(1+k)H^2}\frac{dH}{dk} - \frac{F}{(1+k)^2H}$?
9. AP land: = output/acre F/T	$-\frac{1}{T^2}[F-f_1T-f_2T\frac{dH}{dT}] < 0$	$\frac{1}{T}\frac{dF}{dv} < 0$	$\frac{1}{T}\frac{dF}{dk} < 0$
10. Labor share: $LS = (wL+vH)/F = f_2H/F$	> 0, since $\frac{H}{T}$ falls	> 0, since $\frac{H}{T}$ falls	> 0, since $\frac{H}{T}$ falls
11. Profit: $P = F-wL-vH = F-f_2A$	> 0 then < 0	$-Hf_{22}\frac{dH}{dv} < 0$	$-Hf_{22}\frac{dH}{dk} < 0$
12. Landowner's income: $y = P + wD = Q + wZ$	$\frac{g_{12}Z + (1-a_2g_{22})g_1}{J} > 0$	$Z\frac{dw}{dv} - H < 0$	$Z\frac{dw}{dk} - wH$? *
13. Landowner's consptn: $Q = P + wL = g(T,L)$	$g_1 + g_2k\frac{dH}{dT} > 0$	$(f_2-v)\frac{dH}{dv} - H < 0$	$(f_2-v)\frac{dH}{dk} < 0$
14. Landowner's utility: $U = u(Q,Z)$	$u_1g_1 > 0$	$-u_1H < 0$	$-u_1wH < 0$

* $\frac{dy}{dk} = \frac{f_{22}H(wa_2-Z) - D(f_2-v)}{k^2J}$. The sign is uncertain as long as $\frac{dw}{dk} > 0$.